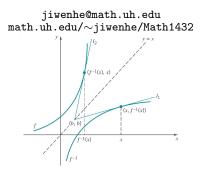
Lecture 1

Section 7.1 One-To-One Functions; Inverses

Jiwen He

Department of Mathematics, University of Houston



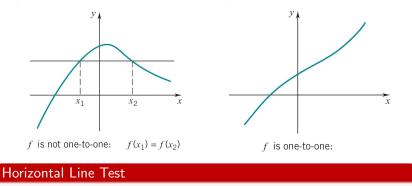


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One-To-One Functions Inverses

Definition Properties Monoton

What are One-To-One Functions? Geometric Test



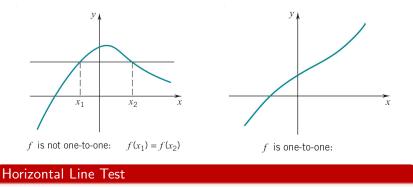
- If some horizontal line intersects the graph of the function more than once, then the function is not one-to-one.
- If no horizontal line intersects the graph of the function more than once, then the function is one-to-one.



One-To-One Functions Inverses

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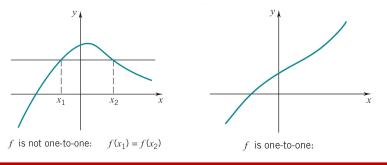


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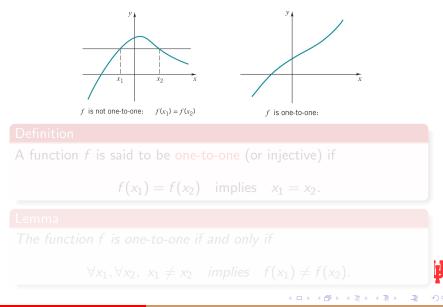
Horizontal Line Test

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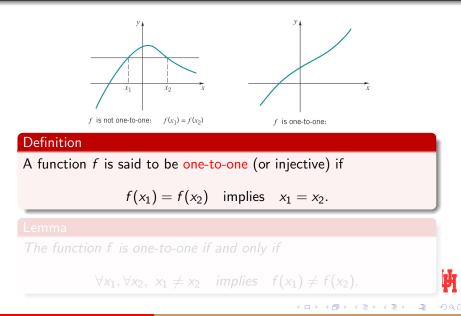
What are One-To-One Functions? Algebraic Test



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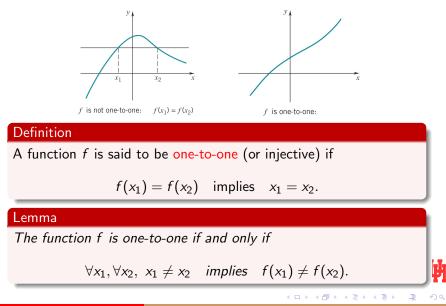
What are One-To-One Functions? Algebraic Test



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What are One-To-One Functions? Algebraic Test



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Examples

- f(x) = 3x 5 is 1-to-1.
- $f(x) = x^2$ is not 1-to-1.
- $f(x) = x^3$ is 1-to-1.
- $f(x) = \frac{1}{x}$ is 1-to-1.
- $f(x) = x^n x$, n > 0, is not 1-to-1.

Proof.

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- $f(x) = x^n x$, n > 0, is not 1-to-1.

Proof.

•
$$f(x_1) = f(x_2) \Rightarrow 3x_1 - 5 = 3x_2 - 5 \Rightarrow x_1 = x_2.$$

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Examples

- f(x) = 3x 5 is 1-to-1.
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- $f(x) = x^n x$, n > 0, is not 1-to-1.

Proof.

• In general, f(x) = ax - b, $a \neq 0$, is 1-to-1.

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Examples

- $f(x) = x^2$ is not 1-to-1.

Proof. H

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Proof.

•
$$f(1) = (1)^2 = 1 = (-1)^2 = f(-1).$$

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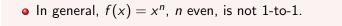
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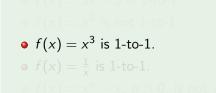
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Examples



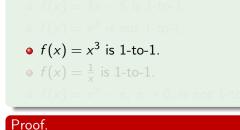
Proof.

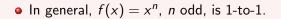
• $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2.$

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Examples







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Examples

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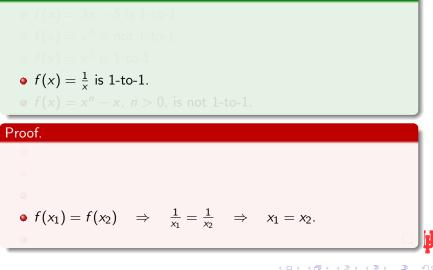
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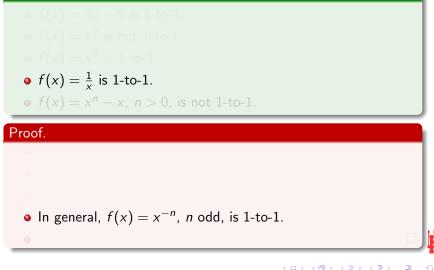
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Examples



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Examples



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Examples

•
$$f(x) = x^n - x$$
, $n > 0$, is not 1-to-1.

Proof.



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Examples



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$$f(x) = x^n - x$$
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Proof.

•
$$f(0) = 0^n - 0 = 0 = (1)^n - 1 = f(1).$$

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Examples • $f(x) = x^n - x$, n > 0, is not 1-to-1. Proof. • In general, 1-to-1 of f and g does not always imply 1-to-1 of f + g. < 白⇒ < Ξ

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Properties

If f and g are one-to-one, then $f \circ g$ is one-to-one.

Examples

•
$$f(x) = 3x^3 - 5$$
 is one-to-one, since $f = g \circ u$ where $g(u) = 3u - 5$ and $u(x) = x^3$ are one-to-one.

•
$$f(x) = (3x - 5)^3$$
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Properties

If f and g are one-to-one, then $f \circ g$ is one-to-one.

Proof.

$$egin{array}{ll} f\circ g(x_1)=f\circ g(x_2)&\Rightarrow&f(g(x_1))=f(g(x_2))&\Rightarrow&g(x_1)=\ g(x_2)&\Rightarrow&x_1=x_2. \end{array}$$

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Properties

If f and g are one-to-one, then $f \circ g$ is one-to-one.

Proof.

$$f \circ g(x_1) = f \circ g(x_2) \Rightarrow f(g(x_1)) = f(g(x_2)) \Rightarrow g(x_1) =$$

 $g(x_2) \Rightarrow x_1 = x_2.$

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 $g(x_2) \Rightarrow x_1 = x_2.$

Examples

$$g(u) = u^3$$
 and $u(x) = 3x - 5$ are one-to-one.

• $f(x) = \frac{1}{3x^3-5}$ is one-to-one, since $f = g \circ u$ where $g(u) = \frac{1}{u}$

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Properties

If f and g are one-to-one, then $f \circ g$ is one-to-one.

Proof.

$$f \circ g(x_1) = f \circ g(x_2) \Rightarrow f(g(x_1)) = f(g(x_2)) \Rightarrow g(x_1) =$$

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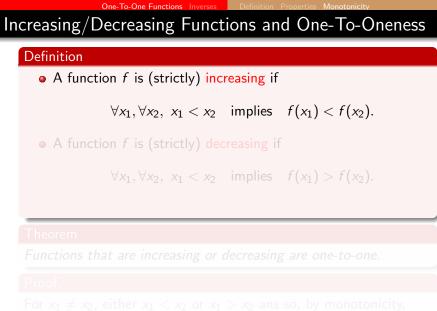
Examples

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For $x_1 \neq x_2$, either $x_1 < x_2$ or $x_1 > x_2$ and so, by monotonicity either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, thus $f(x_1) \neq f(x_2)$.



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Increasing/Decreasing Functions and One-To-Oneness

Definition

• A function f is (strictly) increasing if

 $\forall x_1, \forall x_2, x_1 < x_2 \quad \text{implies} \quad f(x_1) < f(x_2).$

• A function *f* is (strictly) decreasing if

 $\forall x_1, \forall x_2, x_1 < x_2 \text{ implies } f(x_1) > f(x_2).$

Theorem

Functions that are increasing or decreasing are one-to-one.

Proof.

For $x_1 \neq x_2$, either $x_1 < x_2$ or $x_1 > x_2$ and so, by monotonicity, either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, thus $f(x_1) \neq f(x_2)$.



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Theorem

• If f'(x) > 0 for all x, then f is increasing, thus one-to-one.

• If f'(x) < 0 for all x, then f is decreasing, thus one-to-one.

Examples

and

Y'(x) = 0 only at $x = \frac{\pi}{2} + 2k\pi$.



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f'(x) = 0 only at $x = \frac{\pi}{2} + 2k\pi$.



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Examples

•
$$f(x) = x^3 + \frac{1}{2}x$$
 is one-to-one, since
 $f'(x) = 3x^2 + \frac{1}{2} > 0$ for all x.

- $f(x) = -x^5 2x^3 2x$ is one-to-one, since $f'(x) = -5x^4 - 6x^2 - 2 < 0$ for all x
- $f(x) = x \pi + \cos x$ is one-to-one, since $f'(x) = 1 - \sin x \ge 0$

and

$$f'(x) = 0$$
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• $f(x) = -x^5 - 2x^3 - 2x$ is one-to-one, since $f'(x) = -5x^4 - 6x^2 - 2 < 0$ for all x.

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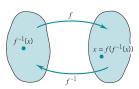
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What are Inverse Functions?



Definition

Let f be a one-to-one function. The inverse of f, denoted by f^{-1} , is the unique function with domain equal to the range of f that satisfies

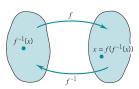
$$f(f^{-1}(x)) = x$$
 for all x in the range of f.

Warning

DON'T Confuse f^{-1} with the reciprocal of f, that is, with 1/f. The "-1" in the notation for the inverse of f is not an exponent; $f^{-1}(x)$ does not mean 1/f(x).



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f⁻¹

Example

•
$$f(x) = x^3 \Rightarrow f^{-1}(x) = x^{1/3}$$

Proof.

• By definition, f^{-1} satisfies the equation

$$f(f^{-1}(x)) = x$$
 for all x .

• Set
$$y = f^{-1}(x)$$
 and solve $f(y) = x$ for y:

$$f(y) = x \quad \Rightarrow \quad y^3 = x \quad \Rightarrow \quad y = x^{1/3}.$$

• Substitute $f^{-1}(x)$ back in for y,

$$f^{-1}(x) = x^{1/3}$$

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f⁻¹

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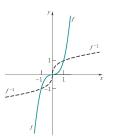
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• Substitute $f^{-1}(x)$ back in for y,

$$f^{-1}(x) = x^{1/3}$$



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Example

•
$$f(x) = x^3 \Rightarrow f^{-1}(x) = x^{1/3}$$
.

In general,

$$f(x) = x^n$$
, *n* odd, $\Rightarrow f^{-1}(x) = x^{1/n}$

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Example

•
$$f(x) = 3x - 5 \Rightarrow f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}$$

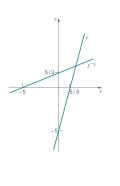
Proof.

By definition, f⁻¹ satisfies f(f⁻¹(x)) = x, ∀x.
Set y = f⁻¹(x) and solve f(y) = x for y:

$$f(y) = x \Rightarrow 3y - 5 = x \Rightarrow y = \frac{1}{3}x + \frac{5}{3}$$

• Substitute $f^{-1}(x)$ back in for y,

$$f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}.$$



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Example

•
$$f(x) = 3x - 5 \Rightarrow f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}$$

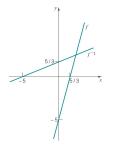
Proof.

- By definition, f^{-1} satisfies $f(f^{-1}(x)) = x$, $\forall x$.
 - Set $y = f^{-1}(x)$ and solve f(y) = x for y

$$f(y) = x \Rightarrow 3y - 5 = x \Rightarrow y = \frac{1}{3}x + \frac{5}{3}$$

• Substitute $f^{-1}(x)$ back in for y,

$$f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}.$$



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Example

•
$$f(x) = 3x - 5 \Rightarrow f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}$$

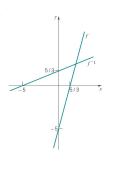
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- Set $y = f^{-1}(x)$ and solve f(y) = x for y:

$$f(y) = x \quad \Rightarrow \quad 3y-5 = x \quad \Rightarrow \quad y = \frac{1}{3}x + \frac{5}{3}.$$

Substitute f⁻¹(x) back in for y,

$$x^{-1}(x) = \frac{1}{3}x + \frac{5}{3}x$$



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Example

•
$$f(x) = 3x - 5 \Rightarrow f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}$$

Proof.

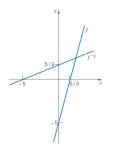
- By definition, f^{-1} satisfies $f(f^{-1}(x)) = x$, $\forall x$.
- Set $y = f^{-1}(x)$ and solve f(y) = x for y:

$$f(y) = x \quad \Rightarrow \quad 3y-5 = x \quad \Rightarrow \quad y = \frac{1}{3}x + \frac{5}{3}.$$

• Substitute $f^{-1}(x)$ back in for y,

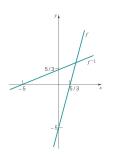
$$f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}.$$

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Example

•
$$f(x) = 3x - 5 \Rightarrow f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}.$$

In general,

$$f(x) = ax + b, \ a \neq 0, \quad \Rightarrow \quad f^{-1}(x) = \frac{1}{a}x - \frac{b}{a}.$$

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Theorem

Proof.

$$f \circ f^{-1} = \mathsf{Id}_{\mathcal{R}(f)}$$
 By definition, f^{-1} satisfies

$$\mathcal{D}(f^{-1}) = \mathcal{R}(f)$$

$$x = f(f^{-1}(\eta))$$

$$f^{-1}$$

$$f^{-1}$$

$$f^{-1}(\eta)$$

 $f(f^{-1}(x)) = x$ for all x in the range of f.

lt is also true that

 $f^{-1}(f(x)) = x$ for all x in the domain of f.

 $f^{-1} \circ f = \operatorname{Id}_{\mathcal{D}(f)}$ $\mathcal{R}(f^{-1}) = \mathcal{D}(f)$



 $f(f^{-1}(y)) = y \qquad \Rightarrow \qquad f(f^{-1}(f(y))) = f(y)$

• f being one-to-one implies $f^{-1}(f(x))) = x$

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Theorem

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 $f^{-1}(x)$



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 $f^{-1}(x)$

$$f^{(x)}$$

$$f^{-1}$$

$$f^{-1}$$

$$f^{-1}$$

$$f^{-1}$$

$$f^{-1}(f(x)) = x$$

 $f^{-1} \circ f = \mathsf{Id}_{\mathcal{D}(f)}$ Proof. $\mathcal{R}(f^{-1}) = \mathcal{D}(f)$ $\forall x \in \mathcal{D}(f), \text{ set } y = f(x).$ Since $y \in \mathcal{R}(f),$

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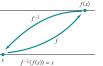
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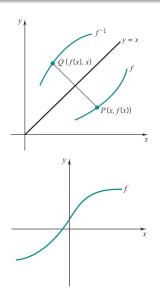
Proof. • $\forall x \in \mathcal{D}(f)$, set y = f(x). Since $y \in \mathcal{R}(f)$, $f(f^{-1}(y)) = y \implies f(f^{-1}(f(x))) = f(x)$.

• f being one-to-one implies
$$f^{-1}(f(x))) = x$$
.

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Graphs of f and f^{-1}

The graph of f^{-1} is the graph of f reflected in the line y = x.

Example

Given the graph of f, sketch the graph of f^{-1} .

Solution

First draw the line y = x. Then reflect the graph of f in that line.

Corollary

f is continuous \Rightarrow so is f $^{-1}$

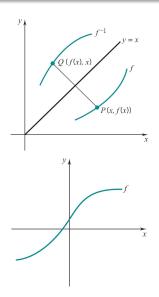
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Graphs of f and f^{-1}

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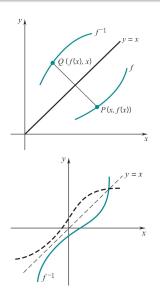
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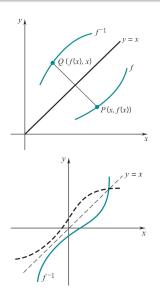
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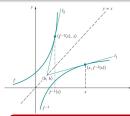
Corollary

f is continuous \Rightarrow

 \Rightarrow so is f^{-1} .

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Differentiability of Inverses



Theorem

$$(f^{-1})'(y) = \frac{1}{f'(x)}, \quad f'(x) \neq 0, \ y = f(x).$$

Proof.

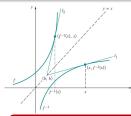
• $\forall y \in \mathcal{D}(f^{-1}) = \mathcal{R}(f), \exists x \in \mathcal{D}(f) \text{ s.t. } y = f(x).$ By definition,

$$f^{-1}(f(x)) = x \quad \Rightarrow \quad \frac{d}{dx}f^{-1}(f(x)) = (f^{-1})'(f(x))f'(x) = 1.$$

• If $f'(x) \neq 0$, then

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)} \quad \Rightarrow \quad (f^{-1})'(y) = \frac{1}{f'(x)}$$

Differentiability of Inverses



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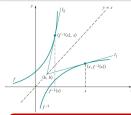
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Example

Let
$$f(x) = x^3 + \frac{1}{2}x$$
. Calculate $(f^{-1})'(9)$.

Solution

- Note that $f'(x) = 3x^2 + \frac{1}{2} > 0$, thus f is one-to-one.
- Note that $(f^{-1})'(y) = \frac{1}{f'(x)}, y = f(x).$
- To calculate (f⁻¹)'(y) at y = 9, find a number x s.t.
 f(x) = 9:

$$f(x) = 9 \quad \Rightarrow \quad x^3 + \frac{1}{2}x = 9 \quad \Rightarrow \quad x = 2.$$

• Since $f'(2) = 3(2)^2 + \frac{1}{2} = \frac{25}{2}$, then $(f^{-1})'(9) = \frac{1}{f'(2)}$



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$$(f^{-1})'(y) = \frac{1}{f'(x)}, y = f(x).$$

• To calculate $(f^{-1})'(y)$ at y = 9, find a number x s.t. f(x) = 9:

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• Since $f'(2) = 3(2)^2 + \frac{1}{2} = \frac{25}{2}$, then $(f^{-1})'(9) = \frac{1}{f'(2)} = \frac{2}{25}$.

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$$(f^{-1})'(y) = \frac{1}{f'(x)}, y = f(x).$$

• To calculate $(f^{-1})'(y)$ at y = 9, find a number x s.t. f(x) = 9:

$$f(x) = 9 \quad \Rightarrow \quad x^3 + \frac{1}{2}x = 9 \quad \Rightarrow \quad x = 2.$$

• Since
$$f'(2) = 3(2)^2 + \frac{1}{2} = \frac{25}{2}$$
, then $(f^{-1})'(9) = \frac{1}{f'(2)} = \frac{2}{25}$.



Example

Let
$$f(x) = x^3 + \frac{1}{2}x$$
. Calculate $(f^{-1})'(9)$.

Note that to calculate $(f^{-1})'(y)$ at a specific y using

$$(f^{-1})'(y) = \frac{1}{f'(x)}, \quad f'(x) \neq 0, \ y = f(x),$$

we only need the value of x s.t. f(x) = y, not the inverse function f^{-1} , which may not be known explicitly.



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Daily Grades

Daily Grades

1.
$$f(x) = x$$
, $f^{-1}(x) = ?$: (a) not exist, (b) x, (c) $\frac{1}{x}$.

2.
$$f(x) = x^3$$
, $f^{-1}(x) = ?$: (a) not exist, (b) $x^{\frac{1}{3}}$, (c) $\frac{1}{x^3}$

3.
$$f(x) = x^2$$
, $f^{-1}(x) = ?$: (a) not exist, (b) $x^{\frac{1}{2}}$, (c) $\frac{1}{x^2}$.

4.
$$f(x) = 3x - 3$$
, $(f^{-1})'(1) = ?$: (a) not exist, (b) 3, (c) $\frac{1}{3}$

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Outline

- One-To-One Functions
 - Definition of the One-To-One Functions
 - Properties of One-To-One Functions
 - Increasing/Decreasing Functions and One-To-Oneness

Inverse Functions

- Definition of Inverse Functions
- Properties of Inverse Functions
- Differentiability of Inverses

