Lecture 1

Section 7.1 One-To-One Functions; Inverses

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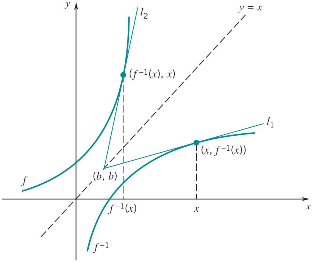
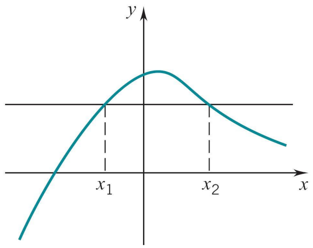
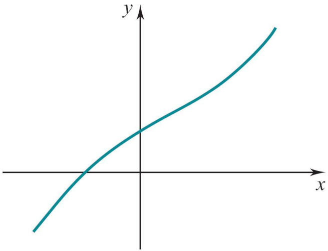


What are One-To-One Functions? Geometric Test



f is not one-to-one: $f(x_1) = f(x_2)$



f is one-to-one:

Horizontal Line Test

- If some horizontal line intersects the graph of the function more than once, then the function is not one-to-one.
- If no horizontal line intersects the graph of the function more than once, then the function is one-to-one.

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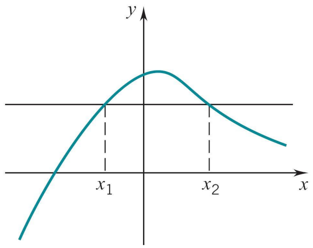

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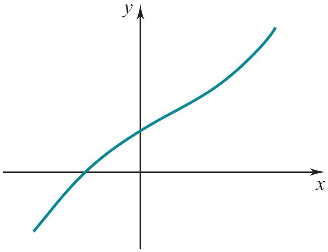

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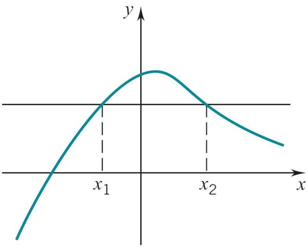

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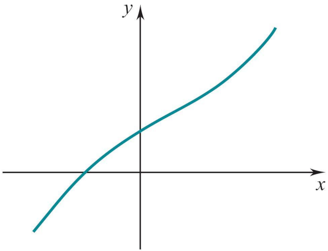

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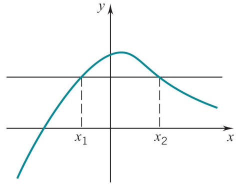
What are One-To-One Functions? Algebraic Test



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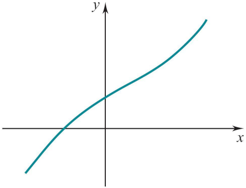

f is one-to-one:

Definition

A function f is said to be one-to-one (or injective) if

$$f(x_1) = f(x_2) \quad \text{implies} \quad x_1 = x_2.$$

Lemma

The function f is one-to-one if and only if

$$\forall x_1, \forall x_2, \ x_1 \neq x_2 \quad \textit{implies} \quad f(x_1) \neq f(x_2).$$

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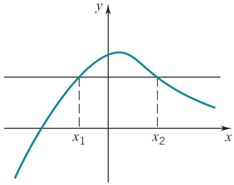

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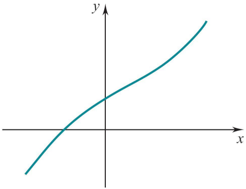

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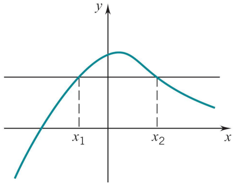

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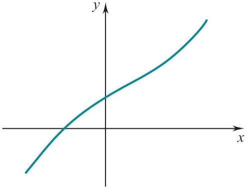

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Examples and Counter-Examples

Examples

- $f(x) = 3x - 5$ is 1-to-1.
- $f(x) = x^2$ is not 1-to-1.
- $f(x) = x^3$ is 1-to-1.
- $f(x) = \frac{1}{x}$ is 1-to-1.
- $f(x) = x^n - x$, $n > 0$, is not 1-to-1.

Proof.

Examples and Counter-Examples

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Proof.

- $f(x_1) = f(x_2) \quad \Rightarrow \quad 3x_1 - 5 = 3x_2 - 5 \quad \Rightarrow \quad x_1 = x_2.$

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- $f(x) = x^n - x$, $n > 0$, is not 1-to-1.

Proof.

- In general, $f(x) = ax - b$, $a \neq 0$, is 1-to-1.

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- $f(x) = x^n - x$, $n > 0$, is not 1-to-1.

Proof.

- $f(1) = (1)^2 = 1 = (-1)^2 = f(-1)$.

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- $f(x) = x^n - x$, $n > 0$, is not 1-to-1.

Proof.

- In general, $f(x) = x^n$, n even, is not 1-to-1.

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- $f(x) = x^n - x$, $n > 0$, is not 1-to-1.

Proof.

- $f(x_1) = f(x_2) \quad \Rightarrow \quad x_1^3 = x_2^3 \quad \Rightarrow \quad x_1 = x_2.$

Examples and Counter-Examples

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- $f(x) = x^n - x$, $n > 0$, is not 1-to-1.

Proof.

- In general, $f(x) = x^n$, n odd, is 1-to-1.

Examples and Counter-Examples

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- $f(x) = x^n - x$, $n > 0$, is not 1-to-1.

Proof.

- $f(x_1) = f(x_2) \quad \Rightarrow \quad \frac{1}{x_1} = \frac{1}{x_2} \quad \Rightarrow \quad x_1 = x_2.$

Examples and Counter-Examples

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- $f(x) = x^n - x$, $n > 0$, is not 1-to-1.

Proof.

- In general, $f(x) = x^{-n}$, n odd, is 1-to-1.

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Proof.

- $f(0) = 0^n - 0 = 0 = (1)^n - 1 = f(1)$. □

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Proof.

- In general, 1-to-1 of f and g does not always imply 1-to-1 of $f + g$. □

Properties

Properties

If f and g are one-to-one, then $f \circ g$ is one-to-one.

Proof.

$f \circ g(x_1) = f \circ g(x_2) \quad \Rightarrow \quad f(g(x_1)) = f(g(x_2)) \quad \Rightarrow \quad g(x_1) = g(x_2) \quad \Rightarrow \quad x_1 = x_2$. □

Examples

- $f(x) = 3x^3 - 5$ is one-to-one, since $f = g \circ u$ where $g(u) = 3u - 5$ and $u(x) = x^3$ are one-to-one.
- $f(x) = (3x - 5)^3$ is one-to-one, since $f = g \circ u$ where $g(u) = u^3$ and $u(x) = 3x - 5$ are one-to-one.
- $f(x) = \frac{1}{3x^3 - 5}$ is one-to-one, since $f = g \circ u$ where $g(u) = \frac{1}{u}$ and $u(x) = 3x^3 - 5$ are one-to-one.

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If f and g are one-to-one, then $f \circ g$ is one-to-one.

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Increasing/Decreasing Functions and One-To-Oneness

Definition

- A function f is (strictly) **increasing** if

$$\forall x_1, \forall x_2,\ x_1 < x_2 \quad \text{implies} \quad f(x_1) < f(x_2).$$

- A function f is (strictly) **decreasing** if

$$\forall x_1, \forall x_2,\ x_1 < x_2 \quad \text{implies} \quad f(x_1) > f(x_2).$$

Theorem

Functions that are increasing or decreasing are one-to-one.

Proof.

For $x_1 \neq x_2$, either $x_1 < x_2$ or $x_1 > x_2$ ans so, by monotonicity, either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, thus $f(x_1) \neq f(x_2)$. □

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Sign of the Derivative Test for One-To-Oneness

Theorem

- *If $f'(x) > 0$ for all x, then f is increasing, thus one-to-one.*
- *If $f'(x) < 0$ for all x, then f is decreasing, thus one-to-one.*

Examples

- $f(x) = x^3 + \frac{1}{2}x$ is one-to-one, since
$$f'(x) = 3x^2 + \frac{1}{2} > 0 \quad \text{for all } x.$$
- $f(x) = -x^5 - 2x^3 - 2x$ is one-to-one, since
$$f'(x) = -5x^4 - 6x^2 - 2 < 0 \quad \text{for all } x.$$
- $f(x) = x - \pi + \cos x$ is one-to-one, since
$$f'(x) = 1 - \sin x \geq 0$$
and
$$f'(x) = 0 \quad \text{only at } x = \frac{\pi}{2} + 2k\pi.$$

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$$f'(x) = 1 - \sin x \geq 0$$
and
$$f'(x) = 0 \quad \text{only at } x = \frac{\pi}{2} + 2k\pi.$$

Sign of the Derivative Test for One-To-Oneness

Theorem

- *If $f'(x) > 0$ for all x, then f is increasing, thus one-to-one.*
- *If $f'(x) < 0$ for all x, then f is decreasing, thus one-to-one.*

Examples

- $f(x) = x^3 + \frac{1}{2}x$ is one-to-one, since
$$f'(x) = 3x^2 + \tfrac{1}{2} > 0 \quad \text{for all } x.$$
- $f(x) = -x^5 - 2x^3 - 2x$ is one-to-one, since
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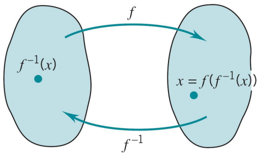
What are Inverse Functions?



Definition

Let f be a one-to-one function. The inverse of f, denoted by f^{-1}, is the unique function with domain equal to the range of f that satisfies

$$f\left(f^{-1}(x)\right) = x \quad \text{for all } x \text{ in the range of } f.$$

Warning

DON'T Confuse f^{-1} with the reciprocal of f, that is, with $1/f$. The "-1" in the notation for the inverse of f is not an exponent; $f^{-1}(x)$ does not mean $1/f(x)$.

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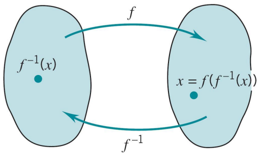

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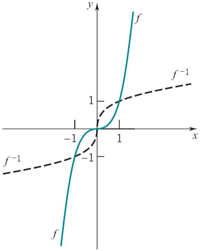

Example

- $f(x) = x^3 \quad \Rightarrow \quad f^{-1}(x) = x^{1/3}.$

Proof.

- By definition, f^{-1} satisfies the equation

$$f(f^{-1}(x)) = x \quad \text{for all } x.$$

- Set $y = f^{-1}(x)$ and solve $f(y) = x$ for y:

$$f(y) = x \quad \Rightarrow \quad y^3 = x \quad \Rightarrow \quad y = x^{1/3}.$$

- Substitute $f^{-1}(x)$ back in for y,

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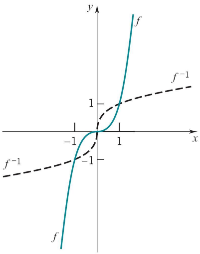
□

Example



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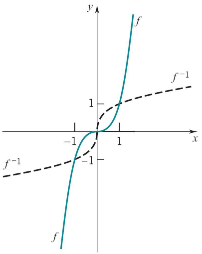

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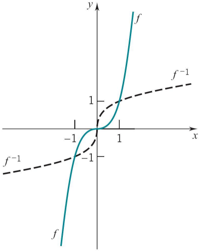

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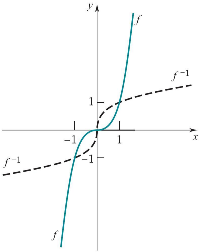

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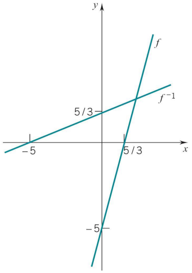
- $f(x) = x^3 \quad \Rightarrow \quad f^{-1}(x) = x^{1/3}.$

In general,

$$f(x) = x^n, \; n \text{ odd}, \quad \Rightarrow \quad f^{-1}(x) = x^{1/n}.$$

Example



Example

- $f(x) = 3x - 5 \quad \Rightarrow \quad f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}$.

Proof.

- By definition, f^{-1} satisfies $f(f^{-1}(x)) = x, \forall x$.
- Set $y = f^{-1}(x)$ and solve $f(y) = x$ for y:

$$f(y) = x \quad \Rightarrow \quad 3y - 5 = x \quad \Rightarrow \quad y = \frac{1}{3}x + \frac{5}{3}.$$

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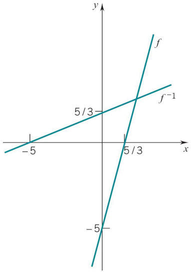

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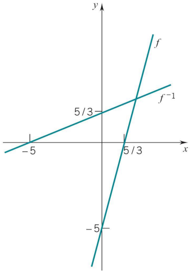

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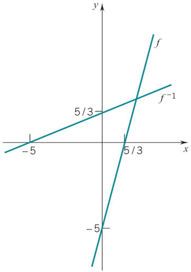

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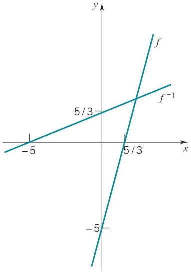

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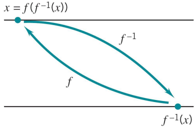
In general,

$$f(x) = ax + b,\ a \neq 0, \quad \Rightarrow \quad f^{-1}(x) = \frac{1}{a}x - \frac{b}{a}.$$

Undone Properties

$$f \circ f^{-1} = \mathrm{Id}_{\mathcal{R}(f)}$$

$$\mathcal{D}(f^{-1}) = \mathcal{R}(f)$$



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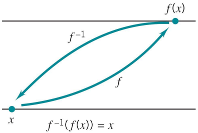

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By definition, f^{-1} satisfies

$$f(f^{-1}(x)) = x \quad \text{for all } x \text{ in the range of } f.$$

It is also true that

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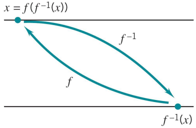

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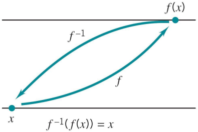

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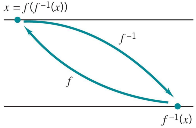

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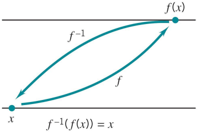

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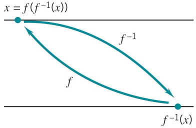

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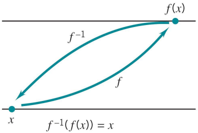

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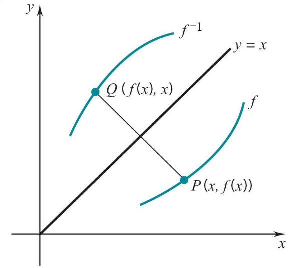
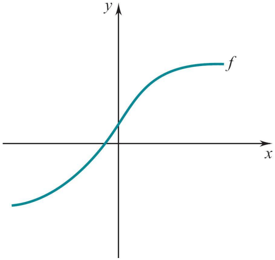
Graphs of f and f^{-1}





Graphs of f and f^{-1}

The graph of f^{-1} is the graph of f reflected in the line $y = x$.

Example

Given the graph of f, sketch the graph of f^{-1}.

Solution

First draw the line $y = x$. Then reflect the graph of f in that line.

Corollary

f is continuous $\Rightarrow$ *so is f^{-1}.*

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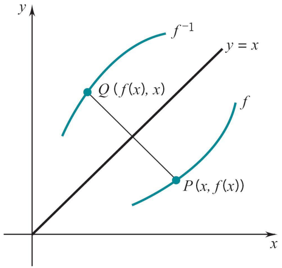

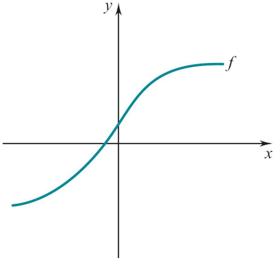

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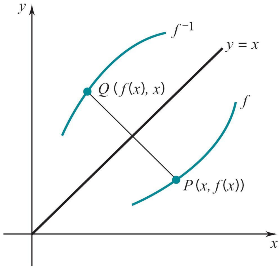

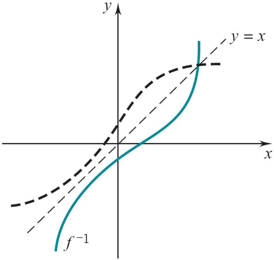

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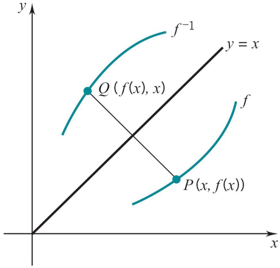

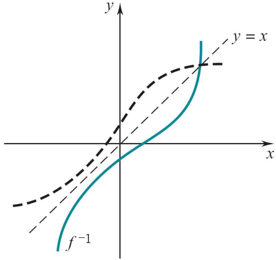

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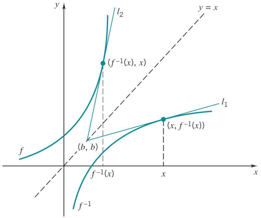
Differentiability of Inverses



Theorem

$$(f^{-1})'(y) = \frac{1}{f'(x)}, \quad f'(x) \neq 0, \; y = f(x).$$

Proof.

- $\forall y \in \mathcal{D}(f^{-1}) = \mathcal{R}(f)$, $\exists x \in \mathcal{D}(f)$ s.t. $y = f(x)$. By definition,

$$f^{-1}(f(x)) = x \quad \Rightarrow \quad \frac{d}{dx} f^{-1}(f(x)) = (f^{-1})'(f(x)) f'(x) = 1.$$

- If $f'(x) \neq 0$, then

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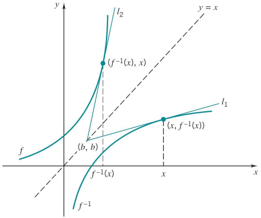

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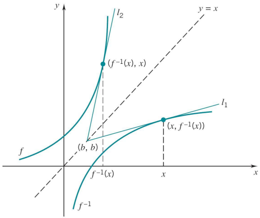

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Let $f(x) = x^3 + \frac{1}{2}x$. Calculate $(f^{-1})'(9)$.

Solution

- Note that $f'(x) = 3x^2 + \frac{1}{2} > 0$, thus f is one-to-one.
- Note that $(f^{-1})'(y) = \frac{1}{f'(x)}$, $y = f(x)$.
- To calculate $(f^{-1})'(y)$ at $y = 9$, find a number x s.t. $f(x) = 9$:

$$f(x) = 9 \quad \Rightarrow \quad x^3 + \frac{1}{2}x = 9 \quad \Rightarrow \quad x = 2.$$

- Since $f'(2) = 3(2)^2 + \frac{1}{2} = \frac{25}{2}$, then $(f^{-1})'(9) = \frac{1}{f'(2)} = \frac{2}{25}$.

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$$f(x) = 9 \quad \Rightarrow \quad x^3 + \frac{1}{2}x = 9 \quad \Rightarrow \quad x = 2.$$

- Since $f'(2) = 3(2)^2 + \frac{1}{2} = \frac{25}{2}$, then $(f^{-1})'(9) = \frac{1}{f'(2)} = \frac{2}{25}$.

Example

Example

Let $f(x) = x^3 + \frac{1}{2}x$. Calculate $(f^{-1})'(9)$.

Solution

- Note that $f'(x) = 3x^2 + \frac{1}{2} > 0$, thus f is one-to-one.
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Example

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Let $f(x) = x^3 + \frac{1}{2}x$. Calculate $(f^{-1})'(9)$.

Note that to calculate $(f^{-1})'(y)$ at a specific y using

$$(f^{-1})'(y) = \frac{1}{f'(x)}, \quad f'(x) \neq 0, \; y = f(x),$$

we only need the value of x s.t. $f(x) = y$, not the inverse function f^{-1}, which may not be known explicitly.

Daily Grades

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1. $f(x) = x,\ f^{-1}(x) =?:$ (a) not exist, (b) x, (c) $\frac{1}{x}$.

2. $f(x) = x^3,\ f^{-1}(x) =?:$ (a) not exist, (b) $x^{\frac{1}{3}}$, (c) $\frac{1}{x^3}$.

3. $f(x) = x^2,\ f^{-1}(x) =?:$ (a) not exist, (b) $x^{\frac{1}{2}}$, (c) $\frac{1}{x^2}$.

4. $f(x) = 3x - 3,\ (f^{-1})'(1) =?:$ (a) not exist, (b) 3, (c) $\frac{1}{3}$.

Outline

- One-To-One Functions
  - Definition of the One-To-One Functions
  - Properties of One-To-One Functions
  - Increasing/Decreasing Functions and One-To-Oneness
- Inverse Functions
  - Definition of Inverse Functions
  - Properties of Inverse Functions
  - Differentiability of Inverses