Lecture 3

Section 7.3 The Logarithm Function, Part II

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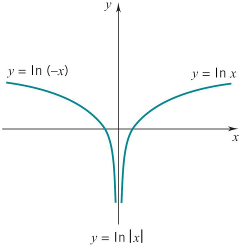
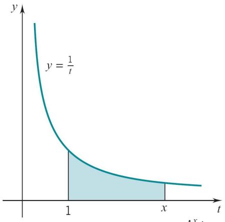


Section 7.2: Highlights



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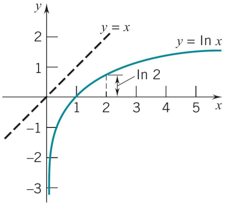

Properties of the Log Function

- $\ln x = \int_1^x \frac{1}{t}\,dt, \qquad \frac{d}{dx}(\ln x) = \frac{1}{x} > 0.$
- $\ln 1 = 0, \qquad \ln e = 1.$
- $\ln(xy) = \ln x + \ln y, \qquad \ln(x/y) = \ln x - \ln y.$
- $\ln(x^r) = r \ln x, \qquad \ln(e^r) = r.$
- domain $= (0, \infty)$, range $= (-\infty, \infty)$.
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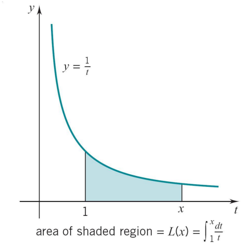

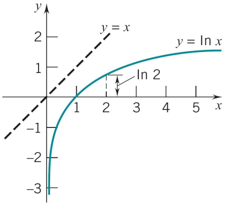

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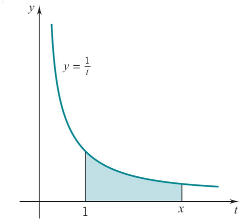

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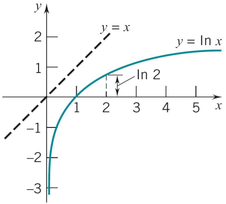

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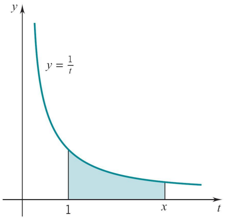

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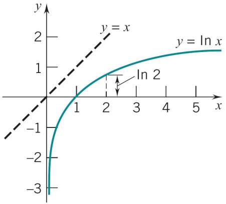

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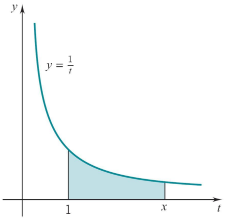

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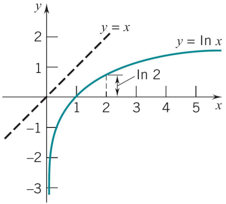

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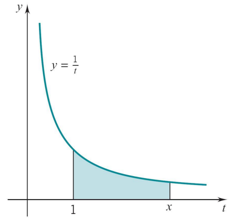

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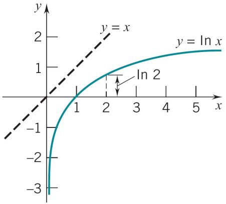

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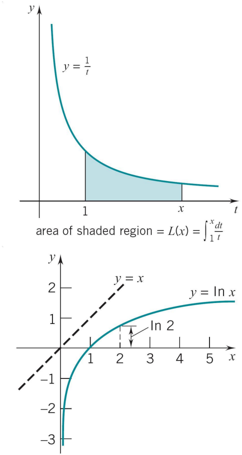

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Differentiation: Chain Rule

Theorem

$$\frac{d}{dx}(\ln u(x)) = \frac{1}{u(x)}\frac{d}{dx}(u(x)), \quad \textit{for } x \textit{ s.t. } u(x) > 0.$$

Proof.

By the chain rule, $\frac{d}{dx}(\ln u) = \frac{d}{du}(\ln u)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$. □

Examples

- $\frac{d}{dx}(\ln(1+x^2)) = \frac{1}{1+x^2}\frac{d}{dx}(1+x^2) = \frac{1}{1+x^2}\cdot 2x = \frac{2x}{1+x^2}$,
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Find the domain of f and find $f'(x)$ if $f(x) = \ln\left(x\sqrt{4+x^2}\right)$.

Solution

- For $x \in \text{domain}(f)$, we need $x\sqrt{4+x^2} > 0$, thus $x > 0$.
- Before differentiating f, simplify it:

$$f(x) = \ln\left(x\sqrt{4+x^2}\right) = \ln x + \ln\left(\sqrt{4+x^2}\right) = \ln x + \frac{1}{2}\ln(4+x^2).$$

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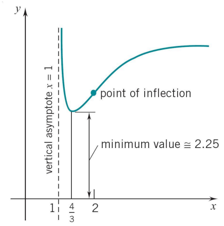

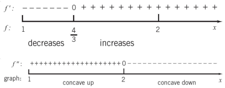

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Let $f(x) = \ln\left(\dfrac{x^4}{x-1}\right)$.

Specify the domain of f.

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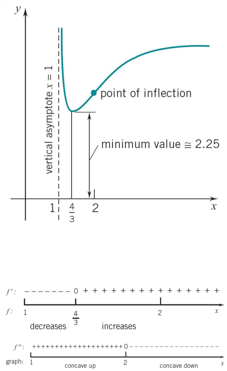

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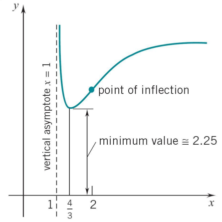

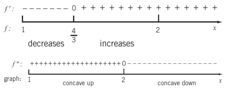

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On what intervals does f increase? Decrease?

Find the extrem values of f.

Solution

- Simplify $f(x) = 4\ln x - \ln(x-1)$. Then
$$f'(x) = \frac{4}{x} - \frac{1}{x-1} = \frac{3x-4}{x(x-1)}.$$
- Thus $f \downarrow$ on $(1, \frac{4}{3})$ and $\uparrow$ on $(\frac{4}{3}, \infty)$.
- At $x = \frac{4}{3}$, $f'(x) = 0$. Thus
$$f(\tfrac{4}{3}) = 4\ln 4 - 3\ln 3 \approx 2.25$$
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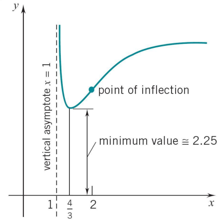

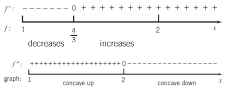

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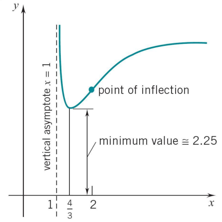

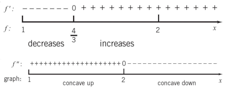

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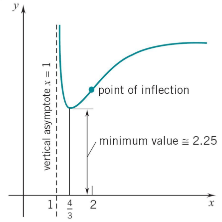

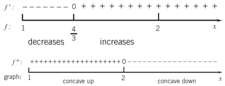

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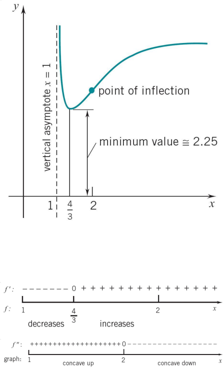

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Determine the concavity and inflection points.

Solution

- From $f'(x) = \frac{4}{x} - \frac{1}{x-1}$, we have
$$f''(x) = -\frac{4}{x^2} + \frac{1}{(x-1)^2} = -\frac{(x-2)(3x-2)}{x^2(x-1)^2}.$$
- At $x = 2$, $f''(x) = 0$ ($\frac{2}{3} \notin \text{domain}(f)$ is ignored). Then, the graph is concave up on $(1,2)$, concave down on $(2,\infty)$.
- The point
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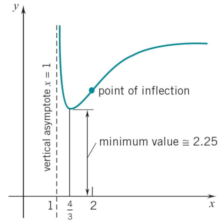

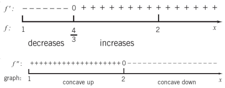

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Determine the concavity and inflection points.

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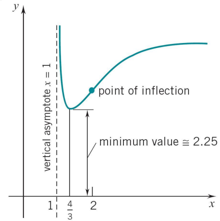

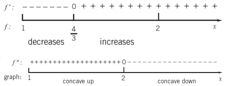

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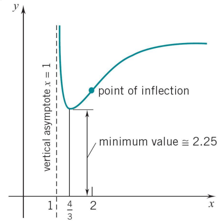

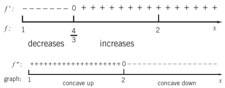

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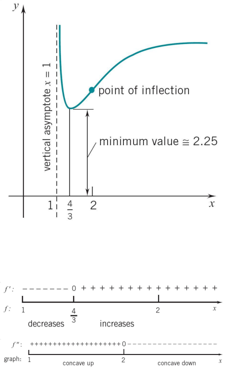

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Sketch the graph, specifying the asymptotes.

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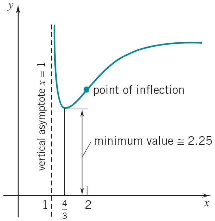

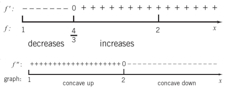

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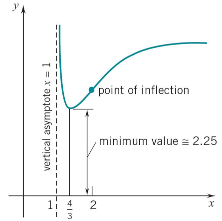

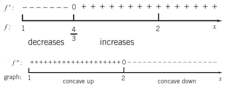

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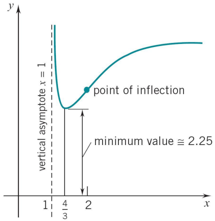

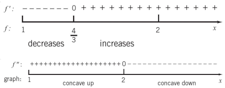

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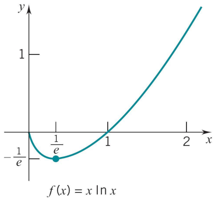

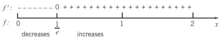

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Specify the domain of f and find the intercepts.

Solution

- $\ln x$ is defined only for $x > 0$, thus

$$\text{domain}(f) = (0, \infty).$$

- There is no y-intercept. Since

$$f(1) = 1 \cdot \ln 1 = 0,$$

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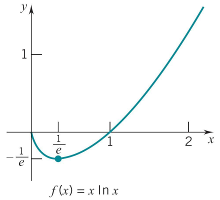

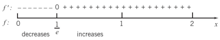

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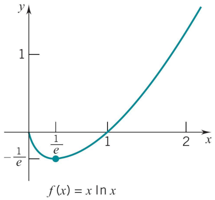

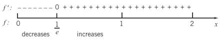

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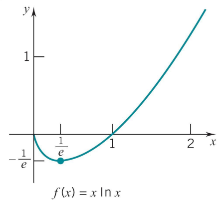

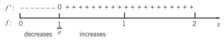

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On what intervals does f increase? Decrease?
Find the extrem values of f.

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  Thus $f \downarrow$ on $(0, \frac{1}{e})$ and $\uparrow$ on $(\frac{1}{e}, \infty)$.
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$$f(\tfrac{1}{e}) = \tfrac{1}{e} \ln \tfrac{1}{e} = -\tfrac{1}{e} \approx -0.368.$$

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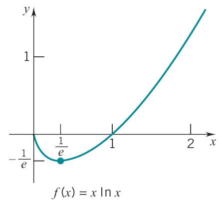

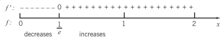

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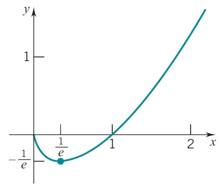

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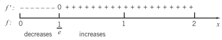

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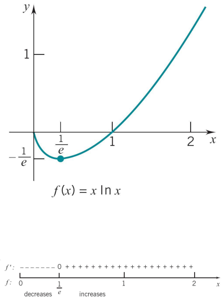

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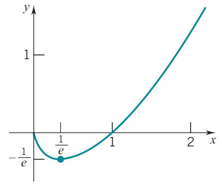

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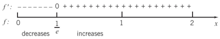

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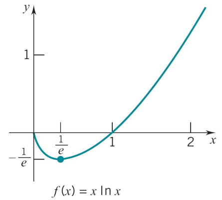

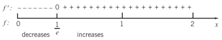

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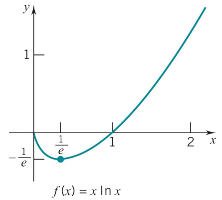

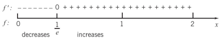

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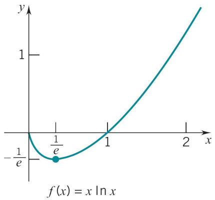

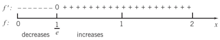

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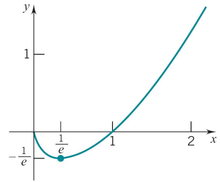

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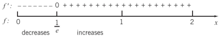

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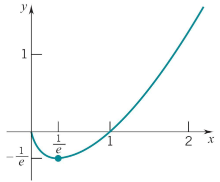

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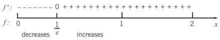

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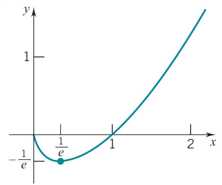

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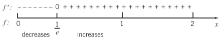

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Quiz

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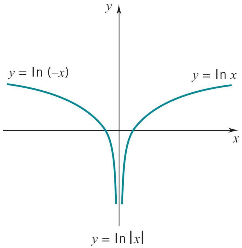
1. $\ln 1 =?$: (a) -1, (b) 0, (c) 1.
2. $\ln e =?$: (a) 0, (b) 1, (c) e.

$f(x) = \ln|x|,\ x \neq 0$



Graph

The graph has two branches:

$$y = \ln(-x),\ x < 0 \text{ and } y = \ln x,\ x > 0,$$

each is the mirror image of the other.

Theorem

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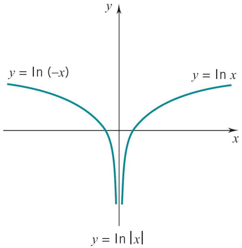

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The graph has two branches:

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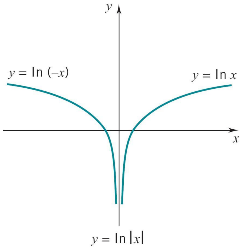

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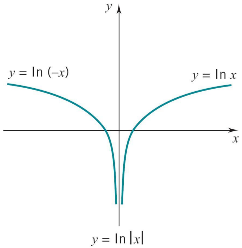

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Power Rule

$$\int x^n\,dx = \begin{cases} \frac{1}{n+1}x^{n+1} + C, & \text{if } n \neq -1, \\ \ln|x| + C, & \text{if } n = -1. \end{cases}$$

Example

$$\int \frac{x+1}{x^2}\,dx = \int \left(\frac{1}{x} + \frac{1}{x^2}\right)\,dx = \ln|x| - \frac{1}{x} + C.$$

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Differentiation: Chain Rule

Theorem

$$\frac{d}{dx}(\ln|u(x)|) = \frac{1}{u(x)}\frac{d}{dx}(u(x)), \quad \textit{for } x \textit{ s.t. } u(x) \neq 0.$$

Proof.

By the chain rule, $\frac{d}{dx}(\ln|u|) = \frac{d}{du}(\ln|u|)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$. □

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- $\frac{d}{dx}(\ln|1-x^3|) = \frac{1}{1-x^3}\frac{d}{dx}(1-x^3) = \frac{-3x^2}{1-x^3}$.
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Logarithmic Differentiation

Theorem

Let $g(x) = g_1(x) \cdot g_2(x) \cdots g_n(x)$. *Then*

$$g'(x) = g(x) \left(\frac{g_1'(x)}{g_1(x)} + \frac{g_2'(x)}{g_2(x)} + \cdots + \frac{g_n'(x)}{g_n(x)} \right).$$

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Find $f'(x)$ if

- $g(x) = x(x-1)(x-2)(x-3)$.
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Quiz (cont.)

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Integration: u-Substitution

Theorem

$$\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C, \quad x \neq 0.$$

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Example

Calculate $\int \frac{x^2}{1-4x^3}dx$.

Let $u = 1 - 4x^3$, thus $du = -12x^2dx$, then

$$\int \frac{x^2}{1-4x^3}dx = -\frac{1}{12}\int \frac{1}{u}du = -\frac{1}{12}\ln|u| + C = -\frac{1}{12}\ln|1-4x^3| + C.$$

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Examples: u-Substitution

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- $\int \frac{\ln x}{x}\, dx.$
- $\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\, dx.$
- $\int_1^2 \frac{6x^2+2}{x^3+x+1}\, dx.$

Solution

Set $u = \ln x, \quad du = \frac{1}{x} dx.$ Then

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Set $u = x^3 + x + 1$, $du = (3x^2+1)dx$. At $x = 1$, $u = 3$; at $x = 2$, $u = 11$. Then

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Solution

- Natural log arises (only) when integrating a quotient whose numerator is the derivative of its denominator (or a constant multiple of it).

$$\int \frac{g'(x)}{g(x)}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|g(x)| + C.$$

Integration of Trigonometric Functions

Recall that

$$\int \cos x\, dx = \sin x + C \quad \Leftrightarrow \quad \frac{d}{dx}\sin x = \cos x.$$
$$\int \sin x\, dx = -\cos x + C \quad \Leftrightarrow \quad \frac{d}{dx}\cos x = -\sin x.$$

$$\int \sec^2 x\, dx = \tan x + C \quad \Leftrightarrow \quad \frac{d}{dx}\tan x = \sec^2 x.$$
$$\int \csc^2 x\, dx = -\cot x + C \quad \Leftrightarrow \quad \frac{d}{dx}\cot x = -\csc^2 x.$$

$$\int \sec x \tan x\, dx = \sec x + C \quad \Leftrightarrow \quad \frac{d}{dx}\sec x = \sec x \tan x.$$
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$$\int \sin x\,dx = -\cos x + C \quad\Leftrightarrow\quad \frac{d}{dx}\cos x = -\sin x.$$

$$\int \sec^2 x\,dx = \tan x + C \quad\Leftrightarrow\quad \frac{d}{dx}\tan x = \sec^2 x.$$

$$\int \csc^2 x\,dx = -\cot x + C \quad\Leftrightarrow\quad \frac{d}{dx}\cot x = -\csc^2 x.$$

$$\int \sec x\,\tan x\,dx = \sec x + C \quad\Leftrightarrow\quad \frac{d}{dx}\sec x = \sec x\,\tan x.$$

$$\int \csc x\,\cot x\,dx = -\csc x + C \quad\Leftrightarrow\quad \frac{d}{dx}\csc x = -\csc x\,\cot x.$$

Integration of Trigonometric Functions

Recall that

$$\int \cos x\,dx = \sin x + C \quad \Leftrightarrow \quad \frac{d}{dx}\sin x = \cos x.$$

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$$\int \csc^2 x\,dx = -\cot x + C \quad \Leftrightarrow \quad \frac{d}{dx}\cot x = -\csc^2 x.$$

$$\int \sec x \tan x\,dx = \sec x + C \quad \Leftrightarrow \quad \frac{d}{dx}\sec x = \sec x \tan x.$$

$$\int \csc x \cot x\,dx = -\csc x + C \quad \Leftrightarrow \quad \frac{d}{dx}\csc x = -\csc x \cot x.$$

Integration of Trigonometric Functions

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$$\int \csc^2 x\,dx = -\cot x + C \quad \Leftrightarrow \quad \frac{d}{dx}\cot x = -\csc^2 x.$$

$$\int \sec x\,\tan x\,dx = \sec x + C \quad \Leftrightarrow \quad \frac{d}{dx}\sec x = \sec x\,\tan x.$$

$$\int \csc x\,\cot x\,dx = -\csc x + C \quad \Leftrightarrow \quad \frac{d}{dx}\csc x = -\csc x\,\cot x.$$

New Integration Formulas

Integration of Trigonometric Functions

$$\int \tan x \, dx = -\ln|\cos x| + C.$$

$$\int \cot x \, dx = \ln|\sin x| + C.$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C.$$

Proof.

Set $u = \cos x$, $du = -\sin x \, dx$, then

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du = -\ln|u| + C$$
$$= -\ln|\cos x| + C.$$

New Integration Formulas

Integration of Trigonometric Functions

$$\int \tan x \, dx = -\ln|\cos x| + C.$$

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Set $u = \cos x$, $du = -\sin x \, dx$, then

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Integration of Trigonometric Functions

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$$\int \csc x \, dx = \ln|\csc x - \cot x| + C.$$

Proof.

Set $u = \sin x$, $du = \cos x \, dx$, then

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{u} du = \ln|u| + C$$
$$= \ln|\sin x| + C.$$

New Integration Formulas

Integration of Trigonometric Functions

$$\int \tan x \, dx = -\ln|\cos x| + C.$$

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Proof.

Set $u = \sin x$, $du = \cos x \, dx$, then

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{1}{u} \, du = \ln|u| + C$$
$$= \ln|\sin x| + C.$$

New Integration Formulas

Integration of Trigonometric Functions

$$\int \tan x \, dx = -\ln|\cos x| + C.$$

$$\int \cot x \, dx = \ln|\sin x| + C.$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C.$$

Proof.

Set $u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)\, dx$, then

$$\int \sec x \, dx = \int \sec \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \, dx$$

$$= \int \frac{1}{u} \, du = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

New Integration Formulas

Integration of Trigonometric Functions

$$\int \tan x \, dx = -\ln|\cos x| + C.$$

$$\int \cot x \, dx = \ln|\sin x| + C.$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

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Proof.

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New Integration Formulas

Integration of Trigonometric Functions

$$\int \tan x \, dx = -\ln|\cos x| + C.$$

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$$\int \csc x \, dx = \ln|\csc x - \cot x| + C.$$

Proof.

Set $u = \csc x - \cot x$, $du = (-\csc x \cot x + \csc^2 x)\, dx$, then

$$\int \csc x \, dx = \int \csc \frac{\csc x - \cot x}{\csc x - \cot x} dx = \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x} dx$$

$$= \int \frac{1}{u} du = \ln|u| + C = \ln|\csc x - \cot x| + C.$$

New Integration Formulas

Integration of Trigonometric Functions

$$\int \tan x\, dx = -\ln|\cos x| + C.$$

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Proof.

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$$\int \csc x\, dx = \int \csc \frac{\csc x - \cot x}{\csc x - \cot x}\, dx = \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x}\, dx$$
$$= \int \frac{1}{u}\, du = \ln|u| + C = \ln|\csc x - \cot x| + C.$$

Examples: $\int \frac{du}{u}$

Examples

- $\displaystyle\int \frac{\sec^2 3x}{1+\tan 3x}\,dx.$
- $\displaystyle\int x \sec x^2\,dx.$
- $\displaystyle\int \frac{\tan(\ln x)}{x}\,dx.$

Solution

Set $u = 1 + \tan 3x,\ du = 3\sec^2 3x\,dx$:

$$\int \frac{\sec^2 3x}{1+\tan 3x}\,dx = \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|1+\tan 3x| + C.$$

Examples: $\int \frac{du}{u}$

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Examples

- $\int \frac{\sec^2 3x}{1+\tan 3x}\,dx.$
- $\int x \sec x^2\,dx.$
- $\int \frac{\tan(\ln x)}{x}\,dx.$

Solution

Set $u = x^2$, $du = 2x\,dx$:

$$\int x \sec x^2\,dx = \frac{1}{2}\int \sec u\,du = \frac{1}{2}\ln|\sec u + \tan u| + C$$
$$= \frac{1}{2}\ln|\sec x^2 + \tan x^2| + C.$$

Examples: $\int \frac{du}{u}$

Examples

- $\int \frac{\sec^2 3x}{1+\tan 3x}\, dx.$
- $\int x \sec x^2 \, dx.$
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Set $u = x^2$, $du = 2x\,dx$:

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$$= \frac{1}{2}\ln|\sec x^2 + \tan x^2| + C.$$

Examples: $\int \frac{du}{u}$

Examples

- $\int \frac{\sec^2 3x}{1 + \tan 3x}\, dx.$
- $\int x \sec x^2 \, dx.$
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Solution

Set $u = x^2$, $du = 2x\, dx$:

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Examples: $\int \frac{du}{u}$

Examples

- $\displaystyle\int \frac{\sec^2 3x}{1+\tan 3x}\,dx.$
- $\displaystyle\int x\sec x^2\,dx.$
- $\displaystyle\int \frac{\tan(\ln x)}{x}\,dx.$

Solution

Set $u = \ln x$, $du = \frac{1}{x}\,dx$:

$$\int \frac{\tan(\ln x)}{x}\,dx = \int \tan u\,du = \ln|\sec u| + C$$
$$= \ln|\sec(\ln x)| + C.$$

Examples: $\int \frac{du}{u}$

Examples

- $\displaystyle\int \frac{\sec^2 3x}{1+\tan 3x}\,dx.$
- $\displaystyle\int x \sec x^2\,dx.$
- $\displaystyle\int \frac{\tan(\ln x)}{x}\,dx.$

Solution

Set $u = \ln x$, $du = \frac{1}{x}\,dx$:

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Examples: $\int \frac{du}{u}$

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Solution

Set $u = \ln x,\ du = \frac{1}{x}\,dx$:

$$\int \frac{\tan(\ln x)}{x}\,dx = \int \tan u\,du = \ln|\sec u| + C$$
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Outline

- Differentiation and Graphing
  - Chain Rule
  - Graphing
- $\ln |x|$
  - Properties
  - Chain Rule
  - Logarithmic Differentiation
- Integration and Trigonometric Functions
  - u-Substitution
  - Trigonometric Functions