Lecture 3 Section 7.3 The Logarithm Function, Part II

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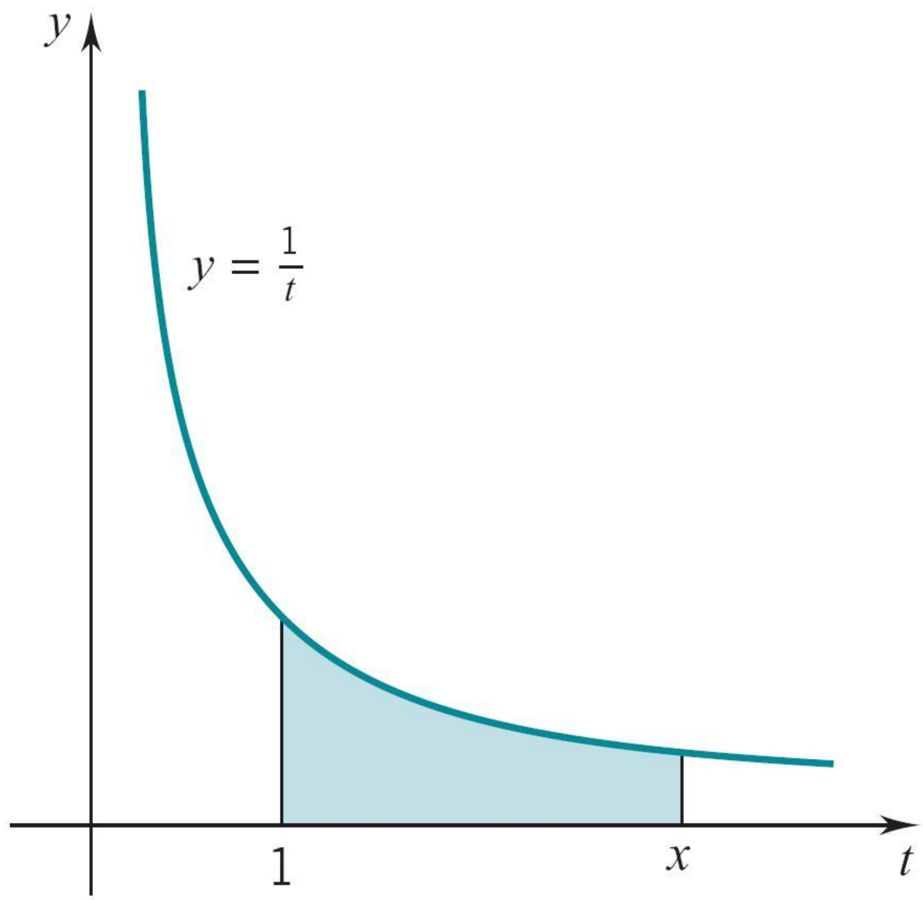
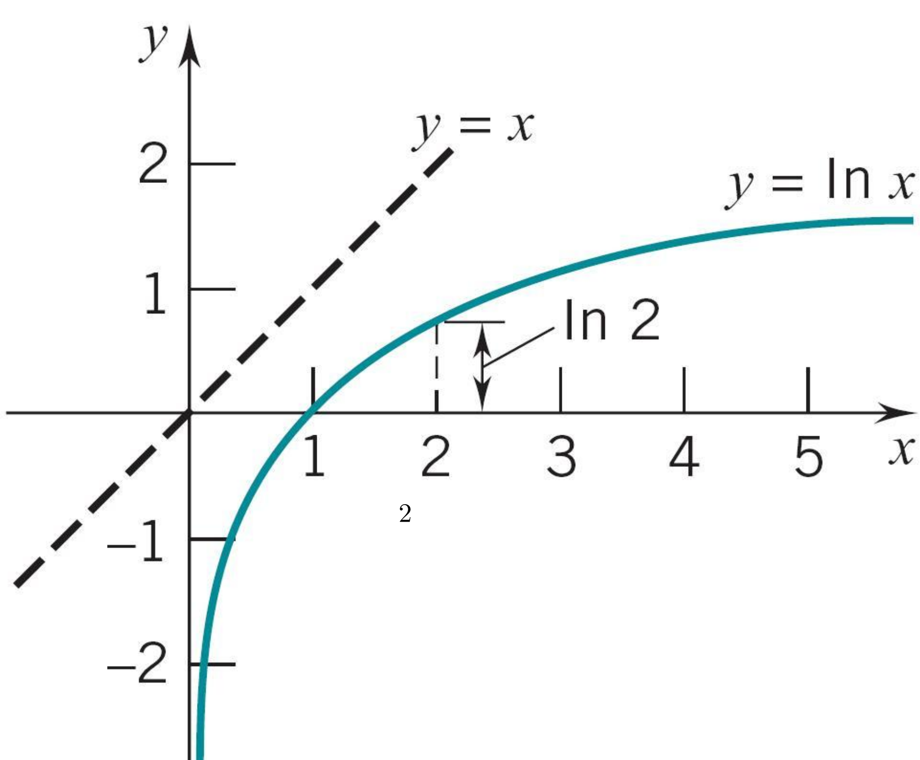
Section 7.2: Highlights



area of shaded region $= L(x) = \int_1^x \frac{dt}{t}$



Properties of the Log Function

- $\ln x = \int_1^x \frac{1}{t}\,dt, \quad \frac{d}{dx}(\ln x) = \frac{1}{x} > 0.$
- $\ln 1 = 0, \quad \ln e = 1.$
- $\ln(xy) = \ln x + \ln y, \quad \ln(x/y) = \ln x - \ln y.$
- $\ln\big(x^r\big) = r\ln x, \quad \ln\big(e^r\big) = r.$
- domain $= (0, \infty)$, range $= (-\infty, \infty)$.
- $\lim_{x\to 0^+} \ln x = -\infty, \quad \lim_{x\to\infty} \ln x = \infty.$
- $\lim_{x\to\infty} \frac{\ln x}{x^r} = 0, \quad \lim_{x\to 0^+} x^r \ln x = 0.$

Limits

1 Differentiation and Graphing

1.1 Chain Rule

Differentiation: Chain Rule

Theorem 1.

$$\frac{d}{dx}\big(\ln u(x)\big) = \frac{1}{u(x)}\frac{d}{dx}\big(u(x)\big), \quad \textit{for } x \textit{ s.t. } u(x) > 0.$$

Proof. By the chain rule, $\frac{d}{dx}\big(\ln u\big) = \frac{d}{du}\big(\ln u\big)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$. □

Examples 2.
- $\frac{d}{dx}\big(\ln(1+x^2)\big) = \frac{1}{1+x^2}\frac{d}{dx}\big(1+x^2\big) = \frac{1}{1+x^2}\cdot 2x = \frac{2x}{1+x^2}$, for all $x \quad \Leftarrow \quad 1+x^2 > 0$.
- $\frac{d}{dx}\big(\ln(1+3x)\big) = \frac{1}{1+3x}\frac{d}{dx}\big(1+3x\big) = \frac{1}{1+3x}\cdot 3 = \frac{3}{1+3x}$, for all $x > -\frac{1}{3}$ $\Leftarrow \quad 1+3x > 0$.

Example

Example 3. Find the domain of f and find $f'(x)$ if $f(x) = \ln\big(x\sqrt{4+x^2}\big)$.

Solution

- For $x \in \text{domain}(f)$, we need $x\sqrt{4+x^2} > 0$, thus $x > 0$.

- Before differentiating f, simplify it:

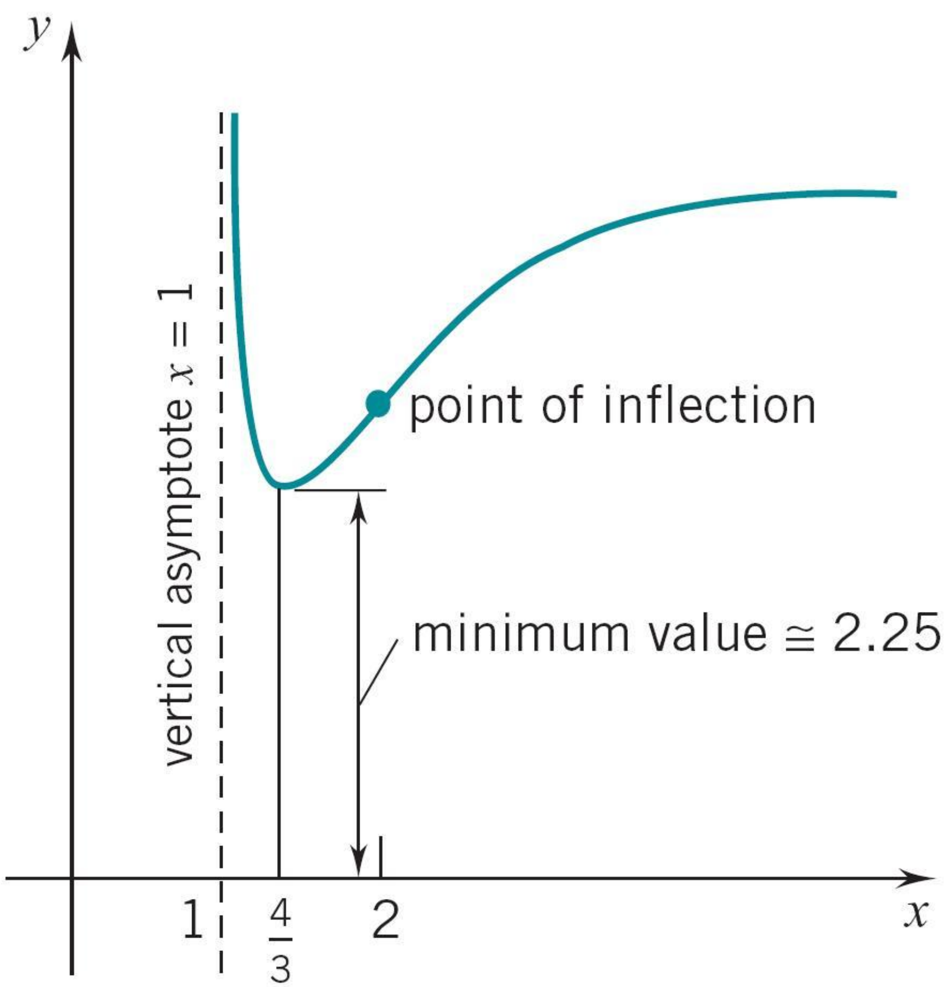
$$f(x) = \ln\bigl(x\sqrt{4+x^2}\bigr) = \ln x + \ln\bigl(\sqrt{4+x^2}\bigr) = \ln x + \frac{1}{2}\ln\bigl(4+x^2\bigr).$$

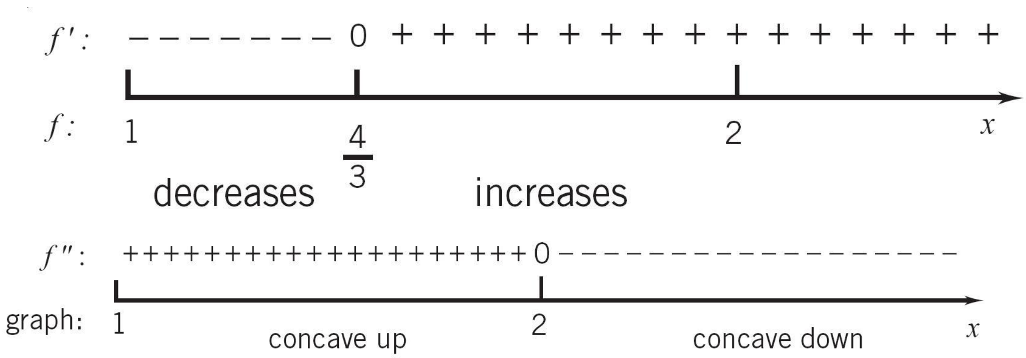
- Thus

$$f'(x) = \frac{1}{x} + \frac{1}{2}\frac{1}{4+x^2}\bigl(4+x^2\bigr)' = \frac{1}{x} + \frac{1}{2}\frac{1}{4+x^2}\cdot 2x = \frac{1}{x} + \frac{x}{4+x^2}.$$

1.2 Graphing

Example





Example 4. Let $f(x) = \ln\left(\frac{x^4}{x-1}\right)$. Specify the domain of f. On what intervals does f increase? Decrease? Find the extrem values of f.Determine the concavity and inflection points. Skectch the graph, specifying the asymptotes.

Solution

For $\frac{x^4}{x-1} > 0$, we need $x > 1$, thus

$$\text{domain}(f) = (1, \infty).$$

- Simplify $f(x) = 4\ln x - \ln(x-1)$. Then $f'(x) = \frac{4}{x} - \frac{1}{x-1} = \frac{3x-4}{x(x-1)}$.
- Thus $f \downarrow$ on $(1, \frac{4}{3})$ and $\uparrow$ on $(\frac{4}{3}, \infty)$.
- At $x = \frac{4}{3}$, $f'(x) = 0$. Thus $f(\frac{4}{3}) = 4\ln 4 - 3\ln 3 \approx 2.25$ is the (only) local and absolute minimum.
- From $f'(x) = \frac{4}{x} - \frac{1}{x-1}$, we have $f''(x) = -\frac{4}{x^2} + \frac{1}{(x-1)^2} = -\frac{(x-2)(3x-2)}{x^2(x-1)^2}$.
- At $x = 2$, $f''(x) = 0$ ($\frac{2}{3} \notin \text{domain}(f)$ is ignored). Then, the graph is concave up on $(1, 2)$, concave down on $(2, \infty)$.
- The point $(2, f(2)) = (2, 4\ln 2) \approx (2, 2.77)$ is the only point of inflection.
- From $f'(x) = \frac{4}{x} - \frac{1}{x-1}$, we have

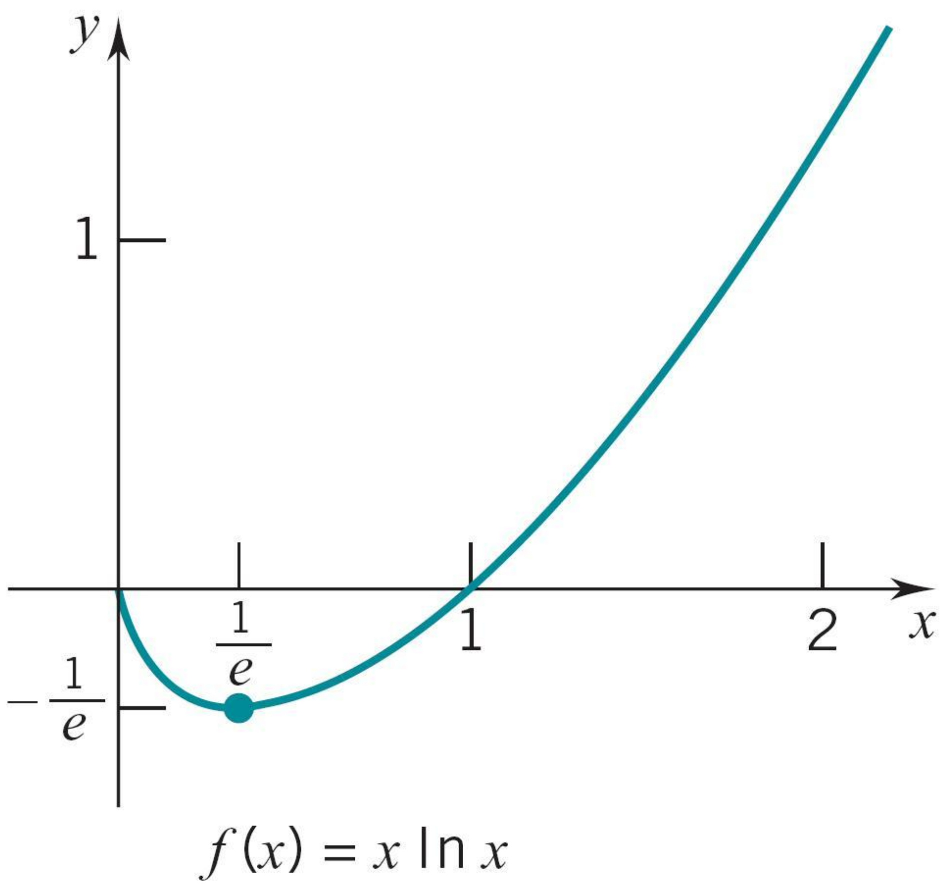
$$\lim_{x\to 1^+} f'(x) = -\infty, \quad \lim_{x\to\infty} f'(x) = 0.$$

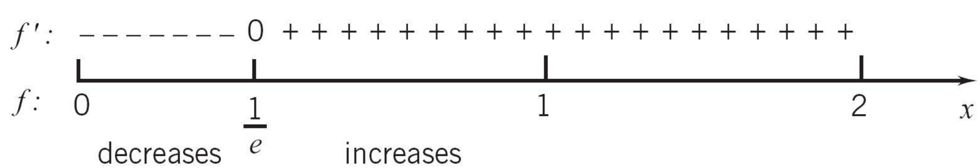
- From $f(x) = 4\ln x - \ln(x-1)$, we have

$$\lim_{x\to 1^+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = \infty.$$

- The line $x = 1$ is a vertical asymptote.

Example





Example 5. Let $f(x) = x \ln x$. Specify the domain of f and find the intercepts. On what intervals does f increase? Decrease? Find the extrem values of f.Determine the concavity and inflection points. Skectch the graph.

Solution

- $\ln x$ is defined only for $x > 0$, thus

$$\text{domain}(f) = (0, \infty).$$

- There is no y-intercept. Since

$$f(1) = 1 \cdot \ln 1 = 0,$$

$x = 1$ is the only x-intercept.

- We have $f'(x) = x \cdot \dfrac{1}{x} + \ln x = 1 + \ln x$.

- For $f'(x) = 0$, we have $$1 + \ln x = 0 \Rightarrow \ln x = -1 \Rightarrow x = \frac{1}{e}.$$ Thus $f \downarrow$ on $(0, \frac{1}{e})$ and $\uparrow$ on $(\frac{1}{e}, \infty)$.

- At $x = \frac{1}{e}$, $f'(x) = 0$. Thus $$f(\tfrac{1}{e}) = \tfrac{1}{e} \ln \tfrac{1}{e} = -\tfrac{1}{e} \approx -0.368.$$ is the (only) local and absolute minimum.

- From $f'(x) = 1+\ln x$, we have $$f''(x) = \frac{1}{x} > 0, \quad \text{for all } x > 0.$$

- Then, the graph is concave up on $(0, \infty)$.

- There is no point of inflection.

- From $f'(x) = 1 + \ln x$, we have

$$\lim_{x \to 0^+} f'(x) = -\infty, \quad \lim_{x \to \infty} f'(x) = \infty.$$

- From $f(x) = x \ln x$, we have

$$\lim_{x \to 0^+} f(x) = 0, \quad \lim_{x \to \infty} f(x) = \infty.$$

Quiz

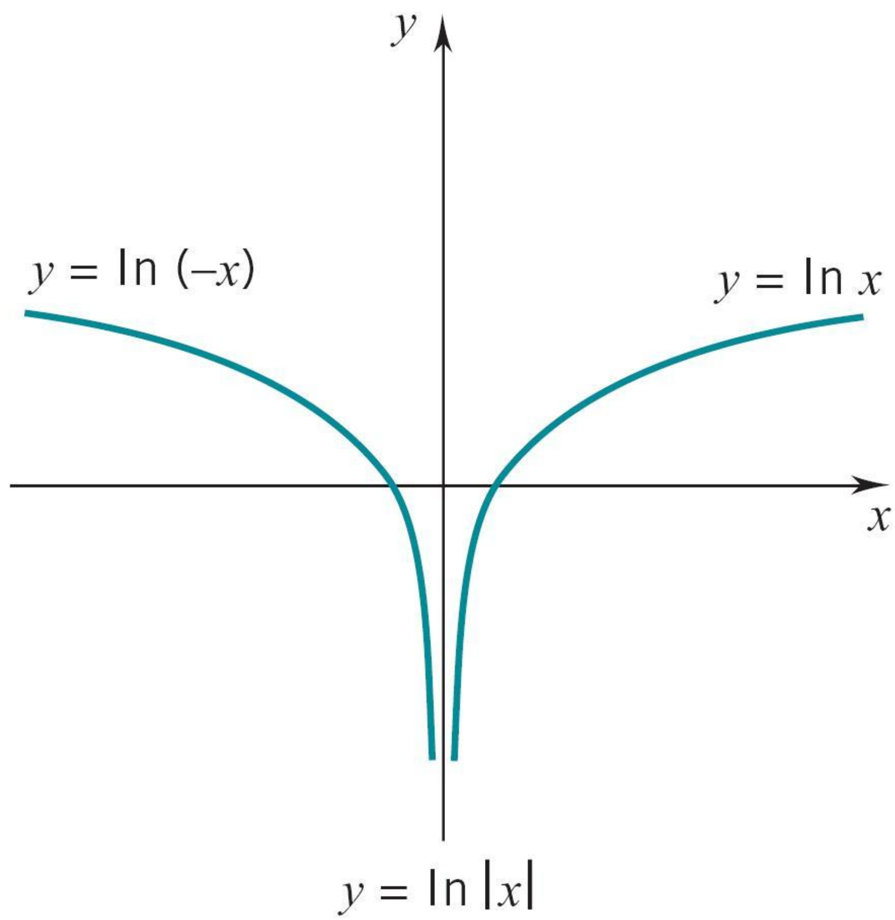
Quiz

1.	$\ln 1 =?$:	(a) -1,	(b) 0,	(c) 1.
2.	$\ln e =?$:	(a) 0,	(b) 1,	(c) e.

2 $\ln |x|$

2.1 Properties

$f(x) = \ln |x|$, $x \neq 0$



Graph
The graph has two branches: $y = \ln(-x)$, $x < 0$ and $y = \ln x$, $x > 0$, each is the mirror image of the other.

Theorem 6.

$$\frac{d}{dx}\big(\ln|x|\big) = \frac{1}{x} \quad \Leftrightarrow \quad \int \frac{1}{x}\,dx = \ln|x| + C$$

Proof. • For $x > 0$, $\dfrac{d}{dx}\big(\ln|x|\big) = \dfrac{d}{dx}\big(\ln x\big) = \dfrac{1}{x}$.

• For $x < 0$, $\dfrac{d}{dx}\big(\ln|x|\big) = \dfrac{d}{dx}\big(\ln(-x)\big) = \dfrac{1}{-x}\dfrac{d}{dx}(-x) = \dfrac{1}{x}$. □

Power Rule: $\int x^n\,dx$

Power Rule

$$\int x^n\,dx = \begin{cases} \dfrac{1}{n+1}x^{n+1} + C, & \text{if } n \neq -1, \\ \ln|x| + C, & \text{if } n = -1. \end{cases}$$

Example 7.

$$\int \frac{x+1}{x^2}\,dx = \int \left(\frac{1}{x} + \frac{1}{x^2}\right)\,dx = \ln|x| - \frac{1}{x} + C.$$

2.2 Chain Rule

Differentiation: Chain Rule

Theorem 8.

$$\frac{d}{dx}\big(\ln|u(x)|\big) = \frac{1}{u(x)}\frac{d}{dx}\big(u(x)\big), \quad \textit{for } x \textit{ s.t. } u(x) \neq 0.$$

Proof. By the chain rule, $\dfrac{d}{dx}\big(\ln|u|\big) = \dfrac{d}{du}\big(\ln|u|\big)\dfrac{du}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$. □

Examples 9.

- $\dfrac{d}{dx}\big(\ln|1-x^3|\big) = \dfrac{1}{1-x^3}\dfrac{d}{dx}\big(1-x^3\big) = \dfrac{-3x^2}{1-x^3}$.
- $\dfrac{d}{dx}\left(\ln\left|\dfrac{x-1}{x-2}\right|\right) = \dfrac{d}{dx}\left(\ln|x-1|\right) - \dfrac{d}{dx}\left(\ln|x-2|\right) = \dfrac{1}{x-1} - \dfrac{1}{x-2}$.

2.3 Logarithmic Differentiation

Logarithmic Differentiation

Theorem 10. *Let* $g(x) = g_1(x) \cdot g_2(x) \cdots g_n(x)$. *Then*

$$g'(x) = g(x)\left(\frac{g_1'(x)}{g_1(x)} + \frac{g_2'(x)}{g_2(x)} + \cdots + \frac{g_n'(x)}{g_n(x)}\right).$$

Proof.

- First write

$$\begin{aligned} \ln|g(x)| &= \ln\left(|g_1(x)| \cdot |g_2(x)| \cdots |g_n(x)|\right) \\ &= \ln|g_1(x)| + \ln|g_2(x)| + \cdots + \ln|g_n(x)|. \end{aligned}$$

- Then differentiate

$$\frac{g'(x)}{g(x)} = \frac{g_1'(x)}{g_1(x)} + \frac{g_2'(x)}{g_2(x)} + \cdots + \frac{g_n'(x)}{g_n(x)}.$$

- Multiplying by $g(x)$ gives the result. □

Examples

Examples 11. Find $f'(x)$ if

- $g(x) = x(x-1)(x-2)(x-3)$.
- $g(x) = \dfrac{(x^2+1)^3(2x-5)^2}{(x^2+5)^2}$.

Solution

$$\ln|g(x)| = \ln|x| + \ln|x-1| + \ln|x-2| + \ln|x-3|.$$

$$\frac{g'(x)}{g(x)} = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}.$$

$$g'(x) = x(x-1)(x-2)(x-3)\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}\right).$$

$$\ln|g(x)| = 3\ln|x^2+1| + 2\ln|2x-5| - 2\ln|x^2+5|.$$

$$\frac{g'(x)}{g(x)} = 3\frac{2x}{x^2+1} + 2\frac{2}{2x-5} - 2\frac{2x}{x^2+5}.$$

$$g'(x) = \frac{(x^2+1)^3(2x-5)^2}{(x^2+5)^2}\left(\frac{6x}{x^2+1} + \frac{4}{2x-5} - \frac{4x}{x^2+5}\right).$$

Quiz (cont.)

Quiz (cont.)

3. $\lim_{x\to 0^+} \ln x = ?:$ (a) $-\infty$, (b) 0, (c) ∞.

4. $\lim_{x\to \infty} \ln x = ?:$ (a) $-\infty$, (b) 0, (c) ∞.

3 Integration and Trigonometric Functions

3.1 u-Substitution

Integration: u-Substitution
Theorem 12.

$$\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C, \quad x \neq 0.$$

Proof.
Let $u = g(x)$, thus $du = g'(x)dx$, then

$$\int \frac{g'(x)}{g(x)} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|g(x)| + C.$$

Example 13. Calculate $\displaystyle\int \frac{x^2}{1-4x^3} dx$. Let $u = 1 - 4x^3$, thus $du = -12x^2 dx$, then $\displaystyle\int \frac{x^2}{1-4x^3} dx = -\frac{1}{12} \int \frac{1}{u} du = -\frac{1}{12} \ln|u| + C = -\frac{1}{12} \ln|1-4x^3| + C.$

Examples: u-Substitution

Examples 14.

- $\displaystyle\int \frac{\ln x}{x}\, dx.$
- $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\, dx.$
- $\displaystyle\int_1^2 \frac{6x^2+2}{x^3+x+1}\, dx.$

Solution

Set $u = \ln x$, $\quad du = \frac{1}{x} dx$. Then

$$\int \frac{\ln x}{x} dx = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} (\ln x)^2 + C.$$

Set $u = 1 + \sqrt{x}$, $\quad du = \frac{1}{2\sqrt{x}}\, dx$. Then

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\, dx = 2 \int \frac{1}{u} du = 2\ln|u| + C = 2\ln\left(1+\sqrt{x}\right) + C.$$

Set $u = x^3 + x + 1$, $\quad du = (3x^2+1)dx$. At $x = 1$, $u = 3$; at $x = 2$, $u = 11$. Then

$$\int_1^2 \frac{6x^2+2}{x^3+x+1} dx = 2\int_3^{11} \frac{1}{u} du = 2\left[\ln|u|\right]_3^{11} = 2\left(\ln 11 - \ln 3\right).$$

- *Natural log arises* (only) when integrating a quotient whose numerator is the derivative of its denominator (or a constant multiple of it).

$$\int \frac{g'(x)}{g(x)} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|g(x)| + C.$$

3.2 Trigonometric Functions

Integration of Trigonometric Functions

Recall that [1ex] $\int \cos x\,dx \;=\; \sin x \,+\, C \qquad \Leftrightarrow \qquad \frac{d}{dx}\sin x \;=\; \cos x.$ $\int \sin x\,dx \;=\; -\cos x + C \qquad \Leftrightarrow \qquad \frac{d}{dx}\cos x \;=\; -\sin x.$ [1ex] $\int \sec^2 x\,dx \;=\; \tan x + C \qquad \Leftrightarrow \qquad \frac{d}{dx}\tan x \;=\; \sec^2 x.$ $\int \csc^2 x\,dx \;=\; -\cot x + C \qquad \Leftrightarrow \qquad \frac{d}{dx}\cot x = -\csc^2 x.$ [1ex] $\int \sec x\ \tan x\,dx = \sec x + C \qquad \Leftrightarrow \qquad \frac{d}{dx}\sec x = \sec x\ \tan x.$ $\int \csc x\ \cot x\,dx = -\csc x + C \qquad \Leftrightarrow \qquad \frac{d}{dx}\csc x = -\csc x\ \cot x.$

New Integration Formulas
Integration of Trigonometric Functions

$\int \tan x\,dx \;=\; -\ln|\cos x| + C.$ $\int \cot x\,dx \;=\; \ln|\sin x| + C.$ $\int \sec x\,dx \;=\; \ln|\sec x + \tan x| + C.$ $\int \csc x\,dx = \ln|\csc x - \cot x| + C.$

Proof.

Set $u = \cos x,\ du = -\sin x\,dx$, then

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C$$
$$= -\ln|\cos x| + C.$$

Set $u = \sin x,\ du = \cos x\,dx$, then

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$
$$= \ln|\sin x| + C.$$

Set $u = \sec x + \tan x,\ du = (\sec x\ \tan x + \sec^2 x)\,dx$, then

$$\int \sec x\,dx = \int \sec \frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{\sec x\ \tan x + \sec^2 x}{\sec x + \tan x}\,dx$$
$$= \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

Set $u = \csc x - \cot x,\ du = (-\csc x\ \cot x + \csc^2 x)\,dx$, then

$$\int \csc x\,dx = \int \csc \frac{\csc x - \cot x}{\csc x - \cot x}\,dx = \int \frac{-\csc x\ \cot x + \csc^2 x}{\csc x - \cot x}\,dx$$
$$= \int \frac{1}{u}\,du = \ln|u| + C = \ln|\csc x - \cot x| + C.$$

Examples: $\int \frac{du}{u}$

Examples 15. • $\int \frac{\sec^2 3x}{1+\tan 3x}\,dx$.

- $\int x \sec x^2\,dx$.
- $\int \frac{\tan(\ln x)}{x}\,dx$.

Solution

Set $u = 1 + \tan 3x$, $du = 3\sec^2 3x\,dx$:

$$\int \frac{\sec^2 3x}{1+\tan 3x}\,dx = \frac{1}{3}\int \frac{1}{u}\,du$$
$$= \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|1+\tan 3x| + C.$$

Set $u = x^2$, $du = 2x\,dx$:

$$\int x\sec x^2\,dx = \frac{1}{2}\int \sec u\,du = \frac{1}{2}\ln|\sec u + \tan u| + C$$
$$= \frac{1}{2}\ln|\sec x^2 + \tan x^2| + C.$$

Set $u = \ln x$, $du = \frac{1}{x}\,dx$:

$$\int \frac{\tan(\ln x)}{x}\,dx = \int \tan u\,du = \ln|\sec u| + C$$
$$= \ln|\sec(\ln x)| + C.$$

Outline

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