Lecture 4

Section 7.4 The Exponential Function Section 7.5 Arbitrary Powers; Other Bases

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Number e



Definition

The number *e* is defined by

$$\ln e = 1$$

i.e., the unique number at which $\ln x = 1$.

Remark

Let $L(x) = \ln x$ and $E(x) = e^x$ for x rational. Then

$$L \circ E(x) = \ln e^x = x \ln e = x,$$

i.e., E(x) is the inverse of L(x).

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Definition

The exp function $E(x) = e^x$ is the inverse of the log function $L(x) = \ln x$:

$$L \circ E(x) = \ln e^x = x, \quad \forall x.$$

Properties

- In x is the inverse of e^x : $\forall x > 0$, $E \circ L = e^{\ln x} = x$
- $\forall x > 0, y = \ln x \quad \Leftrightarrow \quad e^y = x.$
- graph(e^x) is the reflection of graph(ln x) by line y = x.
- range(E) = domain(L) = $(0, \infty)$, domain(E) = range(L) = $(-\infty, \infty)$.
- $\lim_{x \to -\infty} e^x = 0 \quad \Leftrightarrow \quad \lim_{x \to 0^+} \ln x = -\infty,$ $\lim_{x \to 0^+} e^x = \infty \quad \Leftrightarrow \quad \lim_{x \to 0^+} \ln x = \infty.$



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Lemma

•
$$e^{x+y} = e^x \cdot e^y$$
.

•
$$e^{rx} = (e^x)^r$$
, $\forall r$ rational.

Proof

$$\ln e^{x+y} = x + y = \ln e^x + \ln e^y = \ln (e^x \cdot e^y).$$

Since ln x is one-to-one, then

$$e^{x+y}=e^x\cdot e^y.$$



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$$e^{x+y} = e^x \cdot e^y$$
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• $e^{-x} = \frac{1}{e^x}$.
• $e^{x-y} = \frac{e^x}{e^y}$.
• $e^{r^x} = (e^x)^r$, $\forall r$ rational.
Proof
 $1 = e^0 = e^{x+(-x)} = e^x \cdot e^{-x} \Rightarrow e^{-x} = \frac{1}{e^x}$.



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Proof

$$e^{x-y} = e^{x+(-y)} = e^x \cdot e^{-y} = e^x \cdot \frac{1}{e^y} = \frac{e^x}{e^y}$$



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• $e^{rx} = (e^x)^r$, $\forall r \text{ rational.}$

Proof

• For
$$r = m \in \mathbb{N}$$
, $e^{mx} = e^{x + \cdots + x} = e^{x} + e^{x} = (e^{x})^{m}$.

• For
$$r = \frac{1}{n}$$
, $n \in \mathbb{N}$ and $n \neq 0$,
 $c^{x} = c^{\frac{n}{2}x} = (c^{\frac{1}{2}x})^{n} \Rightarrow c^{\frac{1}{2}x} = c^{\frac{1}{2}x}$

• For *r* rational, let $r = \frac{m}{n}$, $m, n \in \mathbb{N}$ and $n \neq 0$. Then $e^{rx} = e^{\frac{m}{n}x} = \left(e^{\frac{1}{n}x}\right)^m = \left((e^x)^{\frac{1}{n}}\right)^m = (e^x)^{\frac{m}{n}} = (e^x)^r$.

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Proof

• For
$$r = m \in \mathbb{N}$$
, $e^{mx} = e^{x} + \cdots + x = e^{m} = (e^x)^m$.
• For $r = \frac{1}{2}$, $n \in \mathbb{N}$ and $n \neq 0$.

$$e^{\mathsf{X}} = e^{\frac{n}{n}\mathsf{X}} = \left(e^{\frac{1}{n}\mathsf{X}}\right)^n \quad \Rightarrow \quad e^{\frac{1}{n}\mathsf{X}} = \left(e^{\mathsf{X}}\right)^{\frac{1}{n}}.$$

• For r rational, let $r = \frac{m}{n}$, $m, n \in \mathbb{N}$ and $n \neq 0$. Then $e^{rx} = e^{\frac{m}{n}x} = \left(e^{\frac{1}{n}x}\right)^m = \left((e^x)^{\frac{1}{n}}\right)^m = (e^x)^{\frac{m}{n}} = (e^x)^r$.

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Lemma

•
$$\frac{d}{dx}e^x = e^x \Rightarrow \int e^x dx = e^x + C.$$

• $\frac{d}{dx^m}e^x = e^x > 0 \implies E(x) = e^x$ is concave up, increasing, and positive.

Proof

Since
$$E(x) = e^x$$
 is the inverse of $L(x) = \ln x$, then with $y = e^x$,

$$\frac{d}{dx}e^{x} = E'(x) = \frac{1}{L'(y)} = \frac{1}{(\ln y)'} = \frac{1}{\frac{1}{y}} = y = e^{x}.$$



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Proof

First, for m = 1, it is true. Next, assume that it is true for k, then

$$\frac{d^{k+1}}{dx^{k+1}}e^{x} = \frac{d}{dx}\left(\frac{d^{k}}{dx^{k}}e^{x}\right) = \frac{d}{dx}\left(e^{x}\right) = e^{x}.$$

By the axiom of induction, it is true for all positive integer m.

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Definition and Properties Differentiation Integration Arbitrary

 e^{x} : as the series $\sum_{k=0}^{\infty} \frac{x^{k}}{k!}$

(0, 1) (1, 0) (1, 0)

Definition

(Section 11.5)

$$e^{x} = \sum_{k=0}^{\infty} \frac{x^{k}}{k!} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots$$
$$= \lim_{n \to \infty} \left(\sum_{k=0}^{n} \frac{x^{k}}{k!} \right), \quad \forall x \in R.$$

 $(k! = 1 \cdot 2 \cdots k)$

Number e

• $e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots = \lim_{n \to \infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \right).$ • $e \approx 2.71828182845904523536\dots$

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Limit:
$$\lim_{x\to\infty} \frac{e}{x}$$

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Theorem

$$\lim_{x\to\infty}\frac{e^x}{x^n}=\infty,\quad\forall n\in\mathbb{N}.$$

Proof.



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Proof.

Recall that

$$e^{x} = \sum_{k=0}^{\infty} \frac{x^{k}}{k!} = 1 + \frac{x}{1} + \frac{x^{2}}{1 \cdot 2} + \frac{x^{3}}{1 \cdot 2 \cdot 3} + \cdots$$

For large x > 0

$$\frac{p}{!} \Rightarrow \frac{e^{x}}{x^{n}} > \frac{x^{p-n}}{p!}$$

• For p > n, $\lim_{x \to \infty} x^{p-n} = \infty$, then $\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$.

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• For
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, $\lim_{x \to \infty} x^{p-n} = \infty$, then $\lim_{x \to \infty} \frac{e^{-n}}{x^n} = \infty$

 $e^{x} >$

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• For large x > 0,

$$x > \frac{x^p}{p!} \quad \Rightarrow \quad \frac{e^x}{x^n} > \frac{x^{p-1}}{p!}$$

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• For
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Quiz

1. domain of
$$\ln(1+x^2)$$
: (a) $x>1$, (b) $x>-1$, (c) any x

2. domain of $\ln(x\sqrt{4+x^2})$: (a) $x \neq 0$, (b) x > 0, (c) any x.



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Lemma

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}.$$

Proof

By the chain rule,

$$\frac{d}{dx}e^{u} = \frac{d}{du}\left(e^{u}\right)\frac{du}{dx} = e^{u}\frac{du}{dx}$$

Examples



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Examples

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$$\frac{d}{dx}e^{kx} = e^{kx} \cdot k = ke^{kx}$$
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• $\frac{d}{dx}e^{\sqrt{x}}$
• $\frac{d}{dx}e^{-x^2}$

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$$\frac{d}{dx}e^{kx} = e^{kx} \cdot k = ke^{kx}$$
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• $\frac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}} \cdot \frac{d}{dx}\sqrt{x} = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}y = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$
• $\frac{d}{dx}e^{-x^2}$

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Lemma

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}.$$

Proof

By the chain rule,

$$\frac{d}{dx}e^{u} = \frac{d}{du}\left(e^{u}\right)\frac{du}{dx} = e^{u}\frac{du}{dx}$$

Examples

•
$$\frac{d}{dx}e^{kx} = e^{kx} \cdot k = ke^{kx}$$

• $\frac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}} \cdot \frac{d}{dx}\sqrt{x} = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}y = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$
• $\frac{d}{dx}e^{-x^2}$

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Examples



Solution

Simplify it before the differentiation:

$$e^{4\ln x} = \left(e^{\ln x}\right)^4 = x^4 \quad \Rightarrow \quad \frac{d}{dx}e^{4\ln x} = \frac{d}{dx}x^4 = 4x^3$$

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Examples



Solution

By the chain rule,

$$\frac{d}{dx}e^{\sin 2x} = e^{\sin 2x}\frac{d}{dx}\sin 2x = e^{\sin 2x} \cdot 2\cos 2x$$

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Examples

•
$$\frac{d}{dx}e^{\sin 2x}$$

• $\frac{d}{dx}e^{\sin 2x}$
• $\frac{d}{dx}\ln(\cos e^{2x})$.

Solution
By the chain rule,

$$\frac{d}{dx} \ln (\cos e^{2x}) = \frac{1}{\cos e^{2x}} \cdot (-\sin e^{2x}) \cdot \frac{d}{dx} e^{2x} = -2e^{2x} \tan e^{2x}.$$

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Examples

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Graph of $f(x) = e^{-\frac{x^2}{2}}$

Example



Let $f(x) = e^{-\frac{x^2}{2}}$. Determine the symmetry of graph and asymptotes.

Solution



Since $f(-x) = e^{-\frac{(-x)^2}{2}} = e^{-\frac{x^2}{2}} = f(x)$ and $\lim_{x \to \pm \infty} e^{-\frac{(-x)^2}{2}} = 0$, the graph is symmetry w.r.t. the *y*-axis, and the *x*-axis is a horizontal asymptote.



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Graph of $f(x) = e^{-\frac{x^2}{2}}$

Example



et
$$f(x) = e^{-\frac{x^2}{2}}$$
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On what intervals does f increase? Decrease? Find the extrem values of f.

Solution

 f:
 0
 dcraws

 f:
 0
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• We have $f'(x) = e^{-\frac{x^2}{2}}(-x) = -xe^{-\frac{x^2}{2}}.$ • Thus $f \uparrow$ on $(-\infty, 0)$ and \downarrow on $(0, \infty)$. • At x = 0, f'(x) = 0. Thus $f(0) = e^0 = 1$ is the (only) local and absolute maximum.



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graph

Determine the concavity and inflection points.

Solution

- From $f'(x) = -xe^{-\frac{x^2}{2}}$, we have $f''(x) = -e^{-\frac{x^2}{2}} + x^2e^{-\frac{x^2}{2}} = (x^2 - 1)e^{-\frac{x^2}{2}}$
- At x = ±1, f"(x) = 0. Then, the graph is concave up on (-∞, -1) and (1,∞); the graph is concave down on (-1,1).
- The points

$$(\pm 1, f(\pm 1)) = (\pm 1, e^{-1})$$

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Quiz (cont.)

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3.
$$\frac{d}{dx}(\ln |x|) = ?$$
: (a) $\frac{1}{x}$, (b) $\frac{1}{|x|}$, (c) $-\frac{1}{x}$.
4. $\int x^{-1} dx = ?$: (a) $\ln x + C$, (b) $\ln |x| + C$, (c) $x^{-1} + C$.



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Theorem

$$\int e^{g(x)}g'(x)\,dx=e^{g(x)}+C.$$

Proof.

Let
$$u = g(x)$$
, thus $du = g'(x)dx$, then

$$\int e^{g(x)}g'(x) \, dx = \int e^{u} du = e^{u} + C = e^{g(x)} + C$$

Example

Calculate
$$\int xe^{-\frac{x^2}{2}} dx$$
.
Let $u = -\frac{2}{3}$, thus $du = -\frac{2}{3}dx$, thus $\int xe^{-\frac{2}{3}} dx = -\int e^u du$

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Arbitrary Powers: $f(x) = x^r$

Definition

For z irrational, we define
$$x^z = e^{z \ln x}$$
, $x > 0$.

Properties (r and s real numbers)

• For
$$x > 0$$
, $x^{r} = e^{r \ln x}$.
• $x^{r+s} = x^{r} \cdot x^{s}$, $x^{r-s} = \frac{x^{r}}{x^{s}}$, $x^{rs} = (x^{r})^{s}$
• $\frac{d}{dx}x^{r} = rx^{r-1}$, $\Rightarrow \int x^{r} dx = \frac{x^{r+1}}{r+1} + C$, for $r \neq -1$.

Example





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$$\frac{d}{dx} \left(x^2 + 1\right)^{3x} = \frac{d}{dx} e^{3x \ln(x^2 + 1)} = e^{3x \ln(x^2 + 1)} \frac{d}{dx} \left(3x \ln(x^2 + 1)\right)$$
$$= e^{3x \ln(x^2 + 1)} \left(\frac{6x^2}{x^2 + 1} + 3\ln(x^2 + 1)\right)$$

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Other Bases: $f(x) = p^x$, p > 0

Definition

For p > 0, the function

$$f(x) = p^x = e^{x \ln p}$$

is called the exp function with base p.

Properties

$$\frac{d}{dx}p^{x} = p^{x}\ln p \quad \Rightarrow \quad \int p^{x} dx = \frac{1}{\ln p}p^{x} + C, \text{ for } p > 0, p \neq 1$$



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Definition

For p > 0, the function

$$f(x) = \log_p x = \frac{\ln x}{\ln p}$$

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Properties

$$\frac{d}{dx}\log_p x = \frac{1}{x\ln p}.$$



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Outline

- Definition and Properties of the Exp Function
 - Definition of the Exp Function
 - Properties of the Exp Function
 - Another Definition of the Exp Function
- Differentiation and Graphing
 - Chain Rule
 - Graphing
- Integration
 - u-Substitution
- Arbitrary Powers
 - Arbitrary Powers
 - Other Bases

