Lecture 4

Section 7.4 The Exponential Function
Section 7.5 Arbitrary Powers; Other Bases

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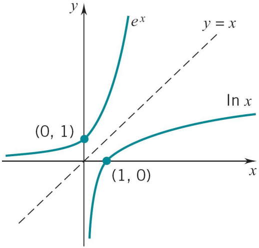
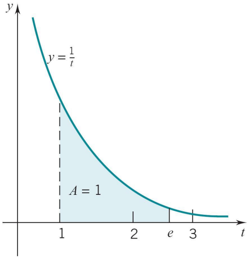


Number e



Definition

The number e is defined by

$$\ln e = 1$$

i.e., the unique number at which $\ln x = 1$.

Remark

Let $L(x) = \ln x$ and $E(x) = e^x$ for x rational. Then

$$L \circ E(x) = \ln e^x = x \ln e = x,$$

i.e., $E(x)$ is the inverse of $L(x)$.

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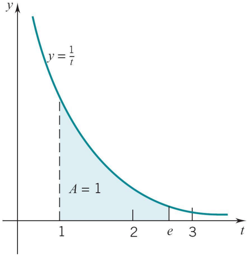

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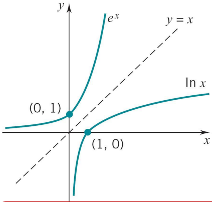
e^x: Inverse of $\ln x$



Definition

The exp function $E(x) = e^x$ is the **inverse** of the log function $L(x) = \ln x$:

$$L \circ E(x) = \ln e^x = x, \quad \forall x.$$

Properties

- $\ln x$ is the inverse of e^x: $\quad \forall x > 0, \quad E \circ L = e^{\ln x} = x.$
- $\forall x > 0,\ y = \ln x \quad \Leftrightarrow \quad e^y = x.$
- graph(e^x) is the reflection of graph($\ln x$) by line $y = x$.
- range(E) = domain(L) = $(0, \infty)$, domain(E) = range(L) = $(-\infty, \infty)$.
- $\lim_{x \to -\infty} e^x = 0 \quad \Leftrightarrow \quad \lim_{x \to 0^+} \ln x = -\infty,$ $\lim_{x \to \infty} e^x = \infty \quad \Leftrightarrow \quad \lim_{x \to \infty} \ln x = \infty.$

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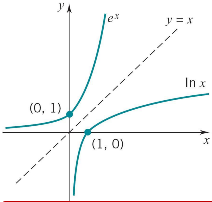

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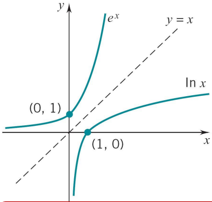

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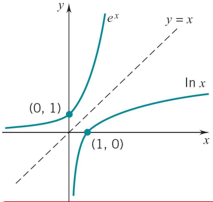

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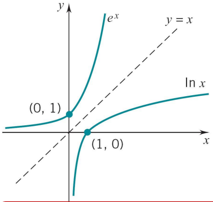

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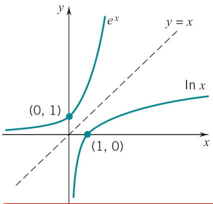

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Algebraic Property

Lemma

- $e^{x+y} = e^x \cdot e^y$.
- $e^{-x} = \dfrac{1}{e^x}$.
- $e^{x-y} = \dfrac{e^x}{e^y}$.
- $e^{rx} = (e^x)^r$, $\forall r$ rational.

Proof

$$\ln e^{x+y} = x + y = \ln e^x + \ln e^y = \ln\left(e^x \cdot e^y\right).$$

Since $\ln x$ is one-to-one, then

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Proof

- For $r = m \in \mathbb{N}$, $e^{mx} = e^{\overbrace{x + \cdots + x}^{m}} = \overbrace{e^x \cdots e^x}^{m} = (e^x)^m$.
- For $r = \frac{1}{n}$, $n \in \mathbb{N}$ and $n \neq 0$,
  $$e^x = e^{\frac{n}{n}x} = \left(e^{\frac{1}{n}x}\right)^n \quad \Rightarrow \quad e^{\frac{1}{n}x} = (e^x)^{\frac{1}{n}}.$$
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Derivatives

Lemma

- $\frac{d}{dx}e^x = e^x \quad \Rightarrow \quad \int e^x \, dx = e^x + C.$
- $\frac{d^m}{dx^m}e^x = e^x > 0 \quad \Rightarrow \quad E(x) = e^x$ *is concave up, increasing, and positive.*

Proof

Since $E(x) = e^x$ is the inverse of $L(x) = \ln x$, then with $y = e^x$,

$$\frac{d}{dx}e^x = E'(x) = \frac{1}{L'(y)} = \frac{1}{(\ln y)'} = \frac{1}{\frac{1}{y}} = y = e^x.$$

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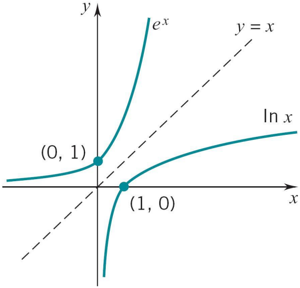
e^x: as the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$



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(Section 11.5)

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{x^k}{k!} \right), \quad \forall x \in R.$$

$(k! = 1 \cdot 2 \cdots k)$

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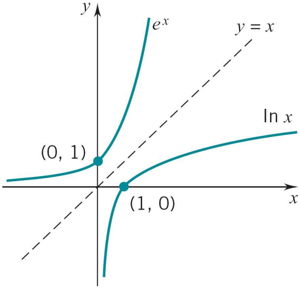

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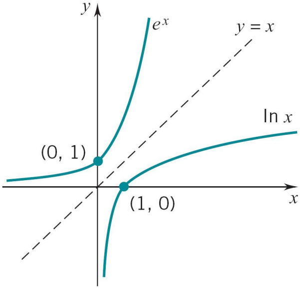

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Limit: $\lim_{x\to\infty} \frac{e^x}{x^n}$

Theorem

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty, \quad \forall n \in \mathbb{N}.$$

Proof.

- Recall that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots .$$

- For large $x > 0$,

$$e^x > \frac{x^p}{p!} \quad \Rightarrow \quad \frac{e^x}{x^n} > \frac{x^{p-n}}{p!}.$$

- For $p > n$, $\lim_{x\to\infty} x^{p-n} = \infty$, then $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$. □

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Limit: $\lim_{x\to\infty} \frac{e^x}{x^n}$

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$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty, \quad \forall n \in \mathbb{N}.$$

Proof.

- Recall that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots .$$

- For large $x > 0$,

$$e^x > \frac{x^p}{p!} \quad \Rightarrow \quad \frac{e^x}{x^n} > \frac{x^{p-n}}{p!}.$$

- For $p > n$, $\lim_{x\to\infty} x^{p-n} = \infty$, then $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$. □

Quiz

Quiz

1. domain of $\ln\left(1+x^2\right)$: (a) $x > 1$, (b) $x > -1$, (c) any x.
2. domain of $\ln\left(x\sqrt{4+x^2}\right)$: (a) $x \neq 0$, (b) $x > 0$, (c) any x.

Differentiation: Chain Rule

Lemma

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

Proof

By the chain rule,

$$\frac{d}{dx}e^u = \frac{d}{du}\left(e^u\right)\frac{du}{dx} = e^u\frac{du}{dx}$$

Examples

- $\frac{d}{dx}e^{kx} = e^{kx}\cdot k = ke^{kx}.$
- $\frac{d}{dx}e^{\sqrt{x}}$
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Examples: Chain Rule

Examples

- $\dfrac{d}{dx} e^{4 \ln x}$.
- $\dfrac{d}{dx} e^{\sin 2x}$.
- $\dfrac{d}{dx} \ln\left(\cos e^{2x}\right)$.

Solution

Simplify it before the differentiation:

$$e^{4 \ln x} = \left(e^{\ln x}\right)^4 = x^4 \quad \Rightarrow \quad \frac{d}{dx} e^{4 \ln x} = \frac{d}{dx} x^4 = 4x^3.$$

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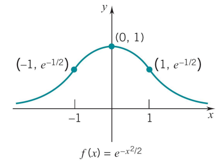
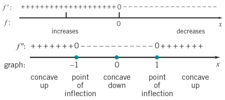
Graph of $f(x) = e^{-\frac{x^2}{2}}$





Example

Let $f(x) = e^{-\frac{x^2}{2}}$.
Determine the symmetry of graph and asymptotes.

Solution

Since $f(-x) = e^{-\frac{(-x)^2}{2}} = e^{-\frac{x^2}{2}} = f(x)$ and $\lim_{x \to \pm\infty} e^{-\frac{(-x)^2}{2}} = 0$, the graph is symmetry w.r.t. the y-axis, and the x-axis is a horizontal asymptote.

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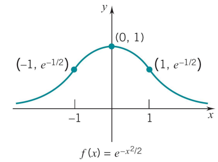

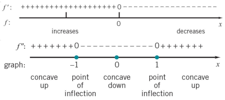

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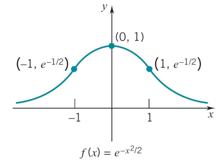

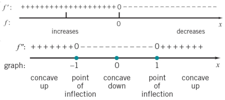

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Let $\quad f(x) = e^{-\frac{x^2}{2}}$.
On what intervals does f increase? Decrease?
Find the extrem values of f.

Solution

- We have
$$f'(x) = e^{-\frac{x^2}{2}}(-x) = -xe^{-\frac{x^2}{2}}.$$
- Thus $f \uparrow$ on $(-\infty, 0)$ and $\downarrow$ on $(0, \infty)$.
- At $x = 0$, $f'(x) = 0$. Thus
$$f(0) = e^0 = 1$$
is the (only) local and absolute maximum.

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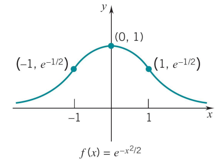

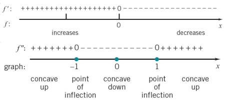

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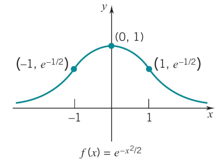

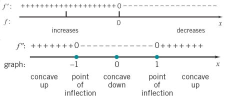

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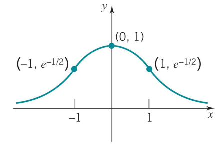

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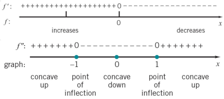

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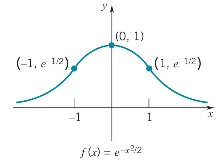

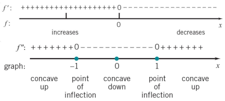

Example

Let $f(x) = e^{-\frac{x^2}{2}}$.

Determine the concavity and inflection points.

Solution

- From $f'(x) = -xe^{-\frac{x^2}{2}}$, we have
$$f''(x) = -e^{-\frac{x^2}{2}} + x^2e^{-\frac{x^2}{2}} = (x^2 - 1)e^{-\frac{x^2}{2}}.$$
- At $x = \pm 1$, $f''(x) = 0$. Then, the graph is concave up on $(-\infty, -1)$ and $(1, \infty)$; the graph is concave down on $(-1, 1)$.
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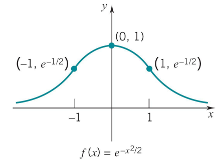

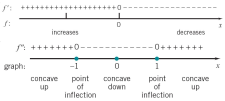

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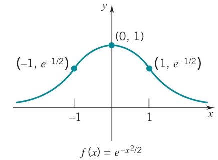

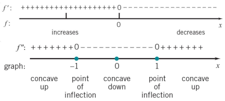

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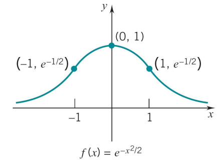

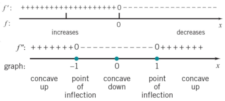

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Quiz (cont.)

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3. $\frac{d}{dx}(\ln|x|) = ?:$ (a) $\frac{1}{x}$, (b) $\frac{1}{|x|}$, (c) $-\frac{1}{x}$.

4. $\int x^{-1}\,dx = ?:$ (a) $\ln x + C$, (b) $\ln|x| + C$, (c) $x^{-1} + C$.

Integration: u-Substitution

Theorem

$$\int e^{g(x)} g'(x)\, dx = e^{g(x)} + C.$$

Proof.

Let $u = g(x)$, thus $du = g'(x)dx$, then

$$\int e^{g(x)} g'(x)\, dx = \int e^u\, du = e^u + C = e^{g(x)} + C.$$

Example

Calculate $\int x e^{-\frac{x^2}{2}}\, dx$.

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Arbitrary Powers: $f(x) = x^r$

Definition

For z irrational, we define $x^z = e^{z \ln x}, \quad x > 0.$

Properties (r and s real numbers)

- For $x > 0$, $x^r = e^{r \ln x}$.
- $x^{r+s} = x^r \cdot x^s, \quad x^{r-s} = \frac{x^r}{x^s}, \quad x^{rs} = (x^r)^s$
- $\frac{d}{dx}x^r = rx^{r-1}, \quad \Rightarrow \quad \int x^r\,dx = \frac{x^{r+1}}{r+1} + C, \text{ for } r \neq -1.$

Example

$$\frac{d}{dx}(x^2+1)^{3x} = \frac{d}{dx}e^{3x\ln(x^2+1)} = e^{3x\ln(x^2+1)}\frac{d}{dx}(3x\ln(x^2+1))$$
$$= e^{3x\ln(x^2+1)}\left(\frac{6x^2}{x^2+1} + 3\ln(x^2+1)\right)$$

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Arbitrary Powers: $f(x) = x^r$

Definition

For z irrational, we define $x^z = e^{z \ln x}, \quad x > 0.$

Properties (r and s real numbers)

- For $x > 0$, $x^r = e^{r \ln x}$.
- $x^{r+s} = x^r \cdot x^s, \quad x^{r-s} = \dfrac{x^r}{x^s}, \quad x^{rs} = (x^r)^s$
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Other Bases: $f(x) = p^x$, $p > 0$

Definition

For $p > 0$, the function

$$f(x) = p^x = e^{x \ln p}$$

is called the exp function with base p.

Properties

$$\frac{d}{dx} p^x = p^x \ln p \quad \Rightarrow \quad \int p^x \, dx = \frac{1}{\ln p} p^x + C, \quad \text{for } p > 0,\ p \neq 1$$

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