Lecture 5

Section 7.6 Exponential Growth and Decay

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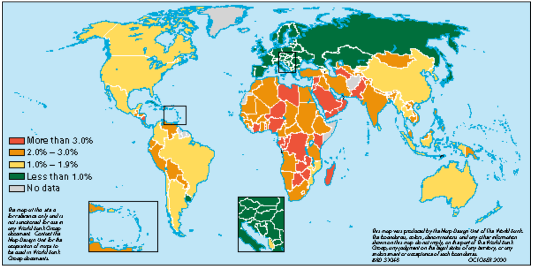
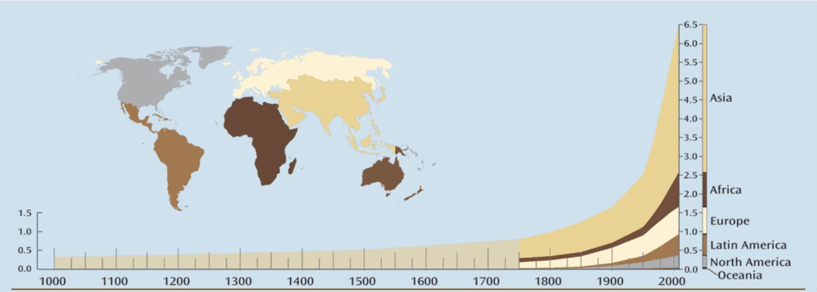


Exponential Growth of the World Population



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- Over the course of human civilization population was fairly stable, growing only slowly until about 1 AD.
- From this point on the population growth accelerated more rapidly and soon reached exponential proportions, leading to more than a quadrupling within the last century.

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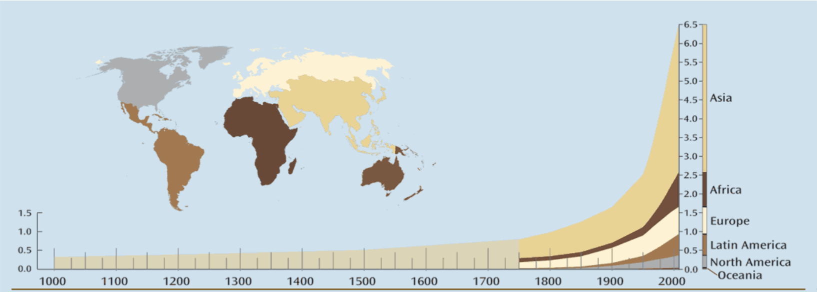

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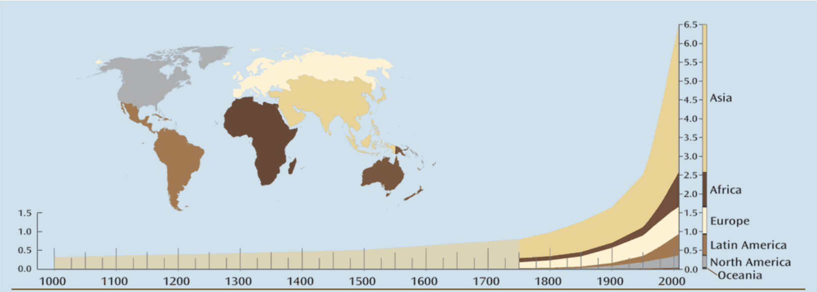

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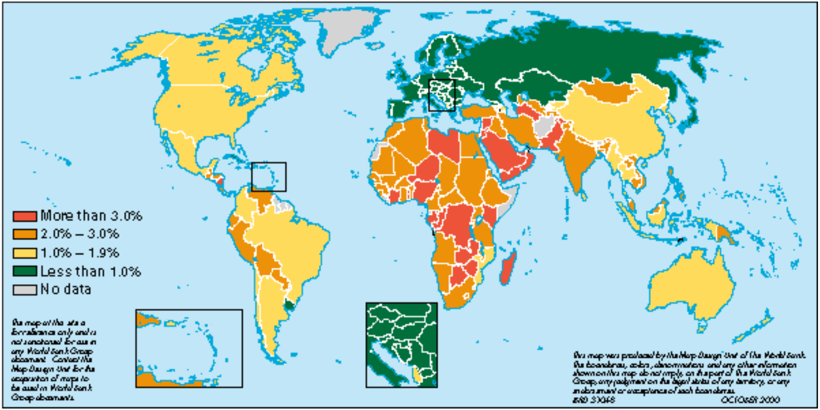
Annual Population Growth Rate 1980-98 (UNESCO)



Annual population growth rate is the increase in a country's population during one year, expressed as a percentage of the population at the start of that period.

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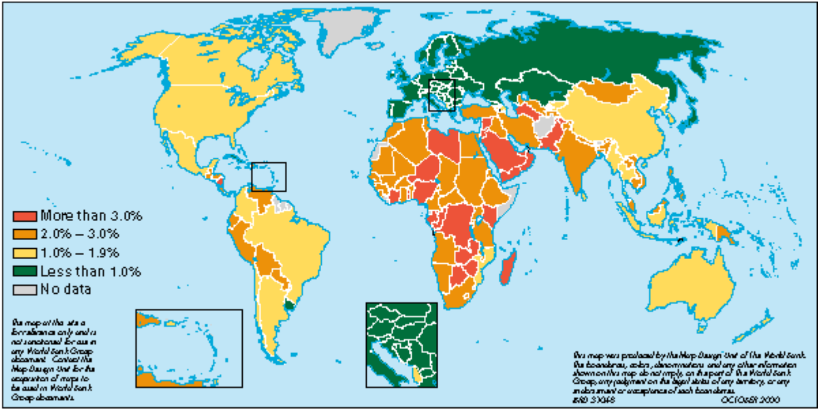

Let $P(t)$ the size of the population P at time t. Then the growth rate

$$k = \frac{P(t+1) - P(t)}{P(t)} \approx \frac{1}{P(t)} \lim_{h \to 0} \frac{P(t+h) - P(t)}{h} = \frac{P'(t)}{P(t)}.$$

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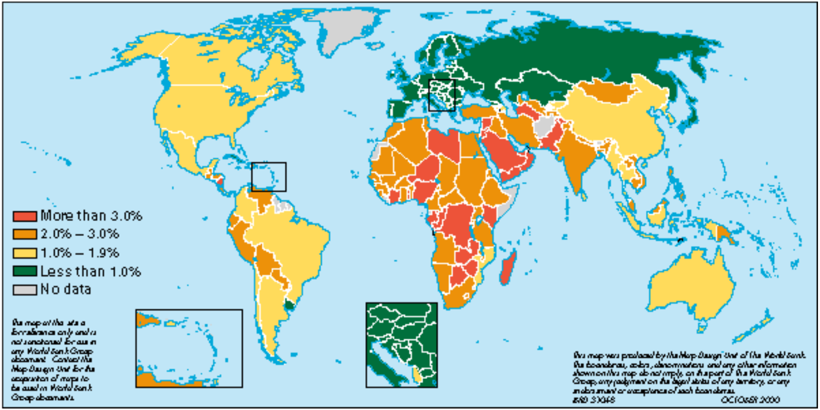

Countries with the most rapid population growth rates tend to be located in Africa and the Middle East.

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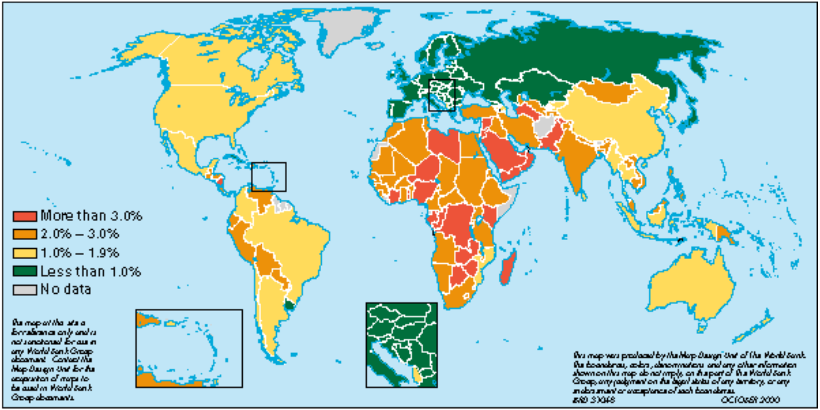
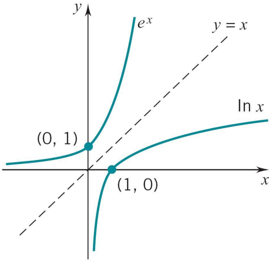


Countries with the slowest population growth rates tend to be located in Europe and North America.

Exponential Growth



Theorem

If

$$f'(t) = kf(t) \quad \text{for all } t \text{ in some interval}$$

then f is an exponential function

$$f(t) = Ce^{kt}$$

where C is arbitrary constant. If the initial value of f at $t = 0$ is known, then

$$C = f(0), \quad f(t) = f(0)e^{kt}$$

Proof.

$$\frac{f'(t)}{f(t)} = k \quad \Rightarrow \quad \frac{d}{dt}\ln f(t) = k$$

$$\ln f(t) = kt + c \quad \Rightarrow \quad f(t) = e^{kt+c} = Ce^{kt}$$

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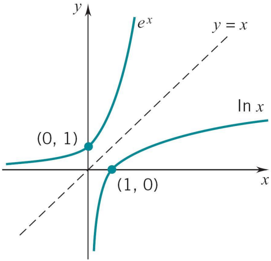

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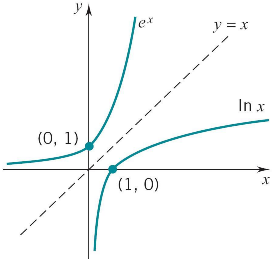

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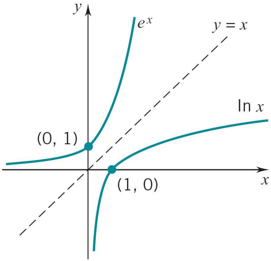

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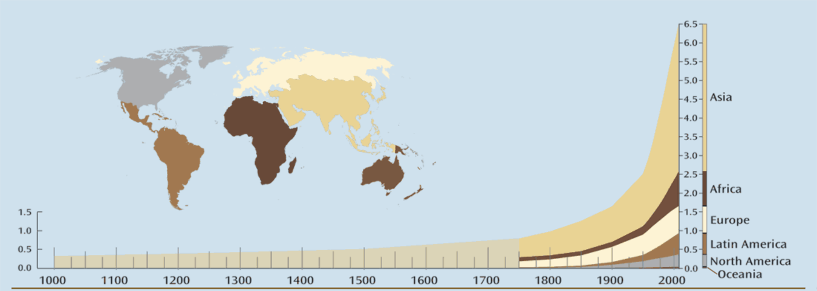

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- The population was 4.5 billion in 1980, and 6 billion in 2000.
- Let $P(t)$ be the population (in billion) t years after 1980 and k the annual population growth rate.
- $P'(t) = kP(t)$ with $P(0) = 4.5$ gives $P(t) = 4.5e^{kt}$.
- $P(20) = 6 \quad \Rightarrow \quad 4.5e^{k20} = 6$, $20k = \ln(6/4.5)$, $k \approx 1.43\%$.

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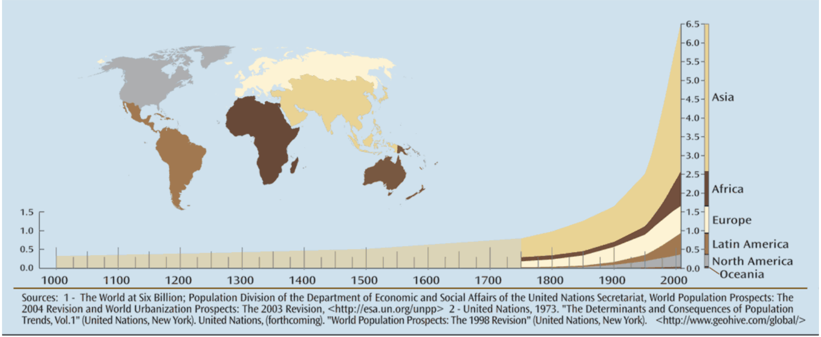

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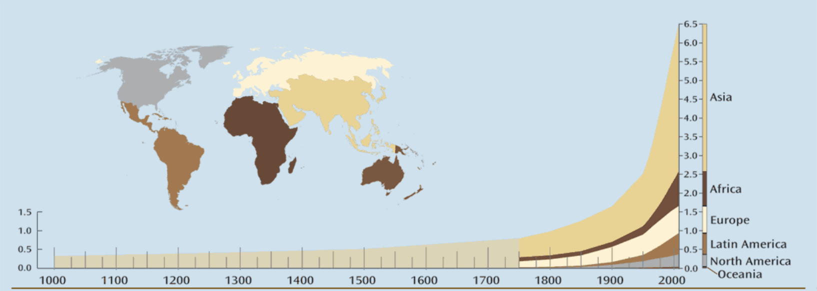

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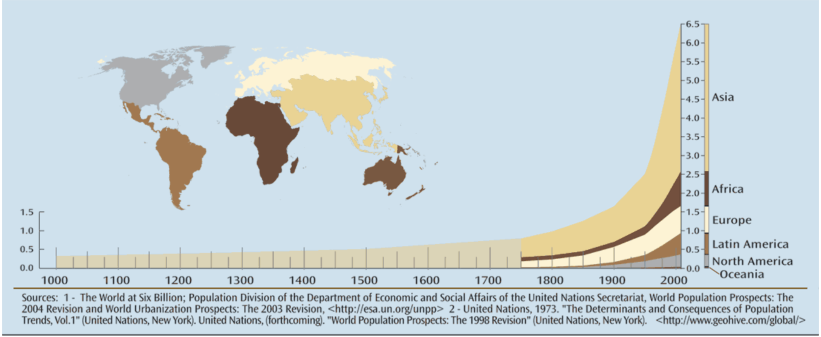

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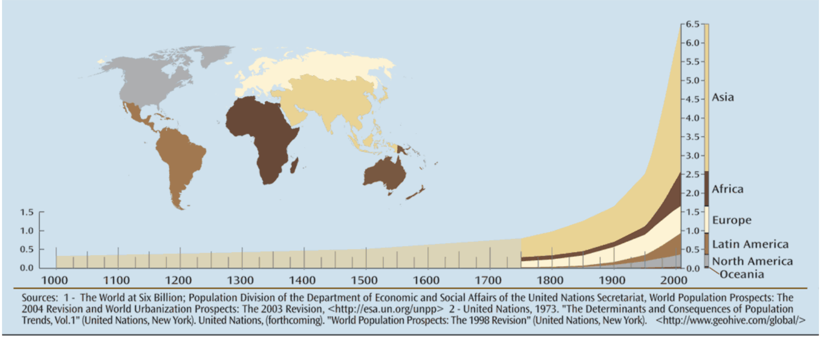

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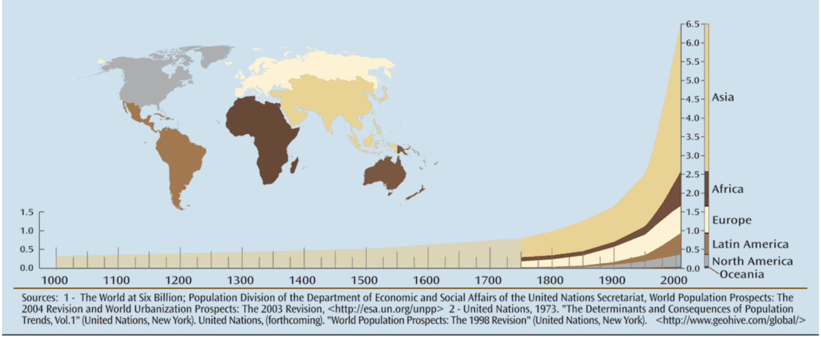

- How long will it take for the population to double from 1980?
- To find the "double time", we solve $2P(0) = P(0)e^{kt}$ for t.
- $e^{kt} = 2,\ kt = \ln 2,\ t = \frac{\ln 2}{k} \approx \frac{0.69}{k\%} = \frac{69}{1.43} \approx 48.5$
- The population will double in 48.5 years (from 1980); that is, the population will reach 9 billion midyear in the year 2028.

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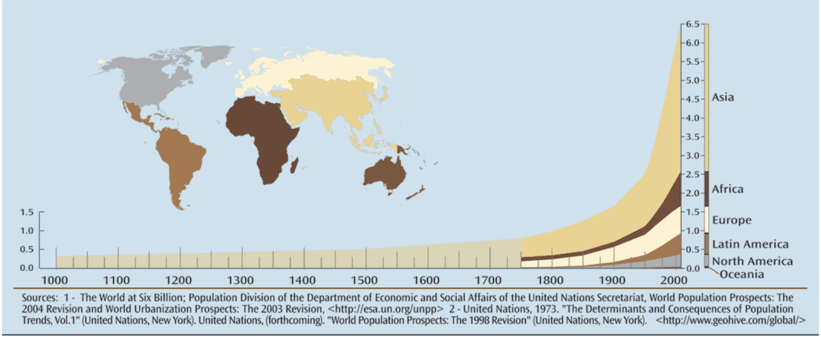

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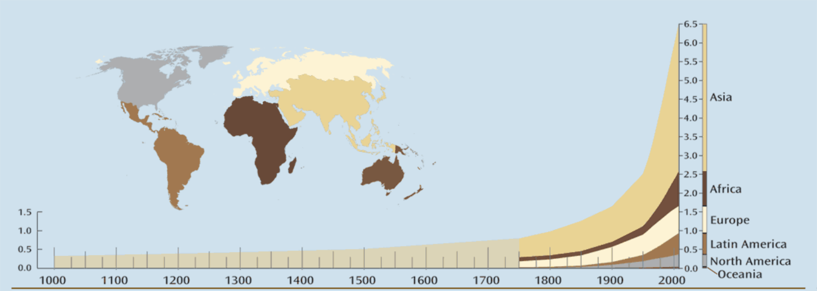

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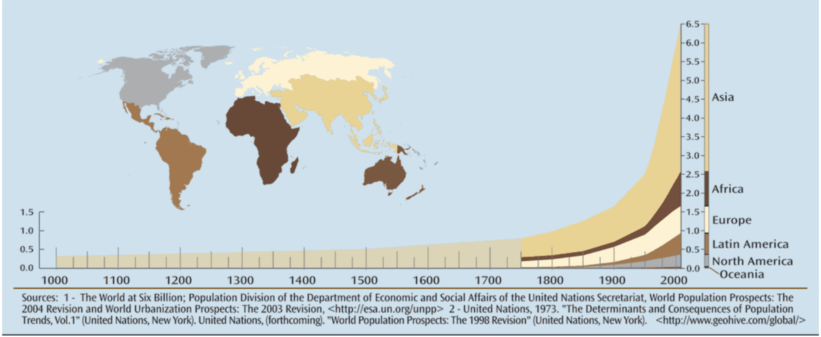

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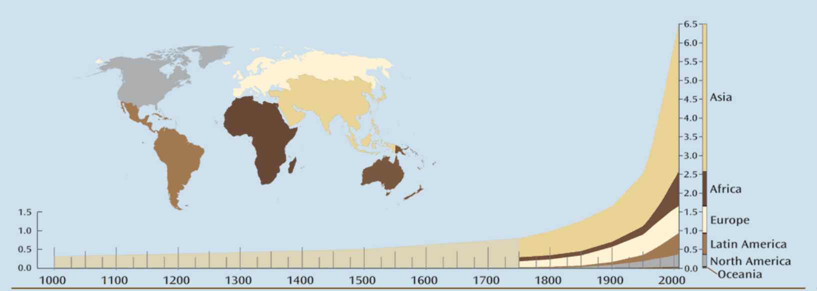

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- Future population projections are notoriously inaccurate: $\frac{d}{dk}\left(P(48.5)\right) = P(0)\frac{d}{dk}\left(e^{48.5k}\right) = 48.5\,P(48.5)$.
- A difference of just 0.1% between predicted and actual growth rates translates into hundreds of millions of lives $48.5 \times 9 \times 0.1\% \approx 0.43$ billion:

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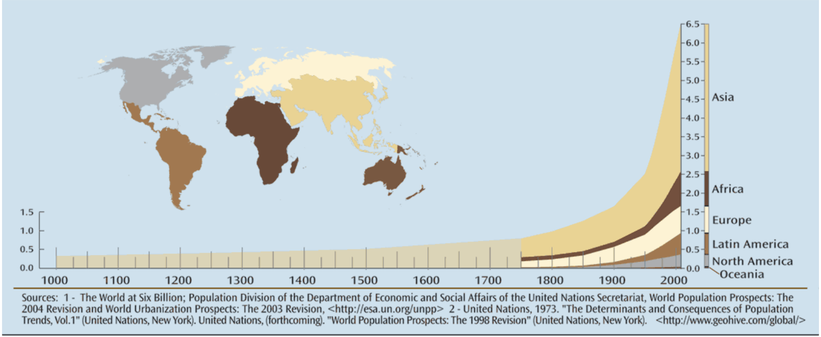

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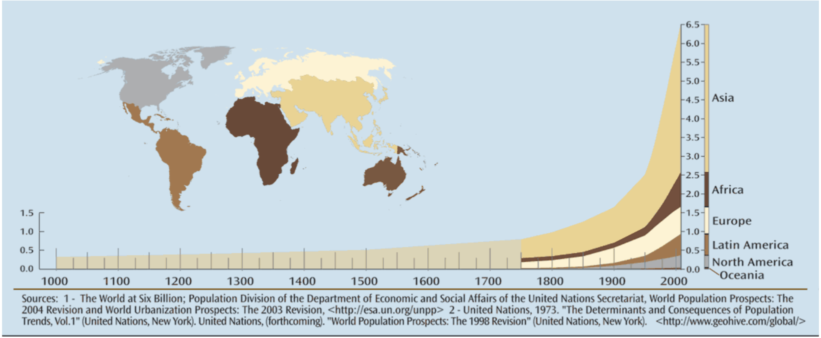

- As of January 1, 2002, demographers were predicting that the world population will peak at 9 billion in the year 2070 and will begin to decline thereafter.
- A sustainable growth of the world population can not be an exponential growth. But what is it?!

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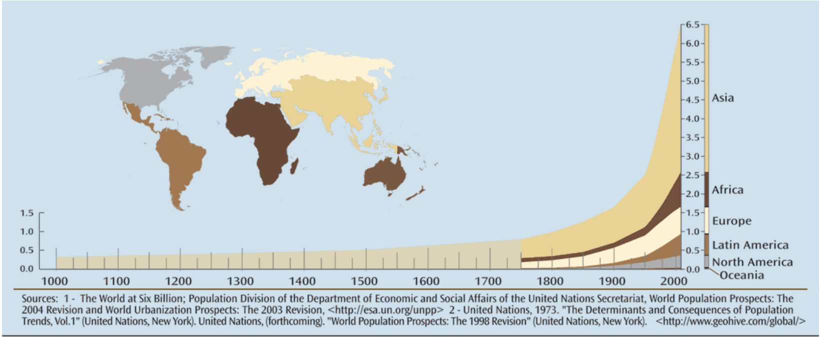

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Quiz

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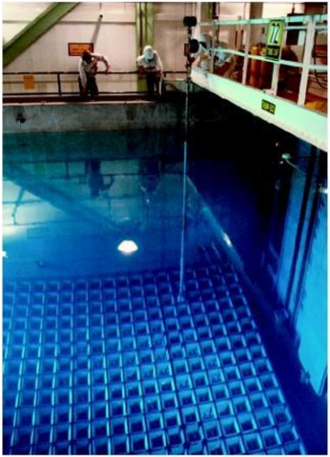
1. $\lim_{x \to -\infty} e^x =:$ (a) 0, (b) $-\infty$, (c) ∞.

2. $\lim_{x \to \infty} e^x =:$ (a) 0, (b) $-\infty$, (c) ∞.

Radioactive Waste



Workers at a nuclear power plant standing near a storage pond filled with spent fuel.

- Radioactive waste (or nuclear waste) is a material deemed no longer useful that has been contaminated by or contains radionuclides.
- Radionuclides are unstable atoms of an element that decay, or disintegrate spontaneously, emitting energy in the form of radiation.
- Radioactive waste has been created by humans as a by-product of various endeavors since the discovery of radioactivity in 1896 by Antoine Henri Becquerel.

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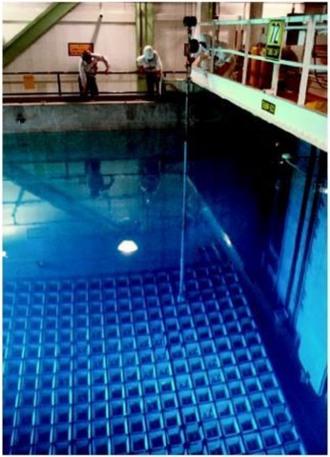

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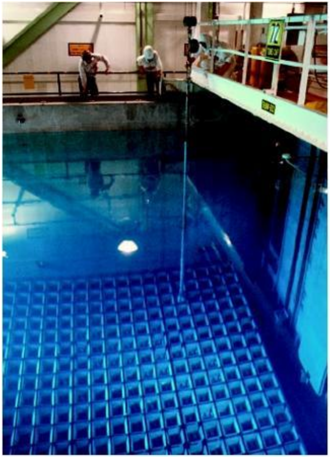

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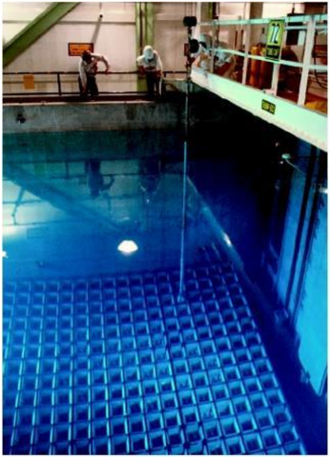

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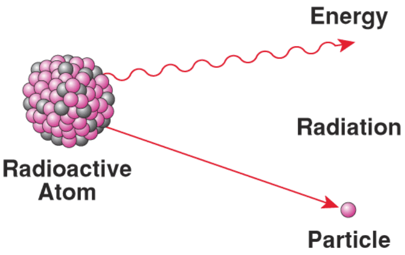
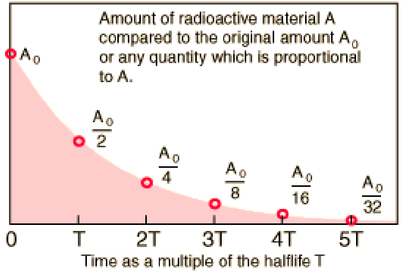
Half-Life of Radioactive Material





- A half-life is a measure of time required for an amount of radioactive material to decrease by one-half of its initial amount.
- The half-life of a radionuclide can vary from fractions of a second to millions of years: sodium-26 (1.07 seconds), hydrogen-3 (12.3 years), carbon-14 (5,730 years), uranium-238 (4.47 billion years).

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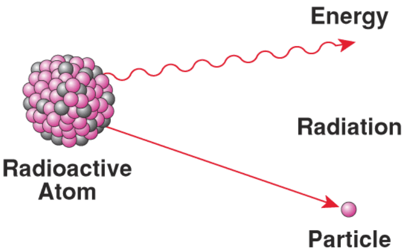

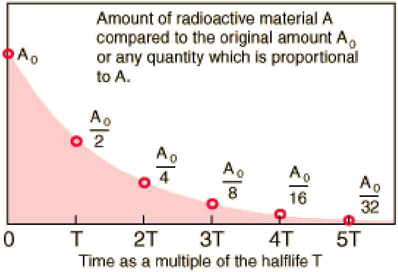

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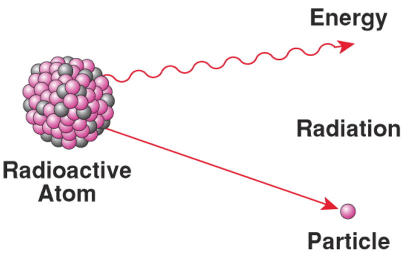

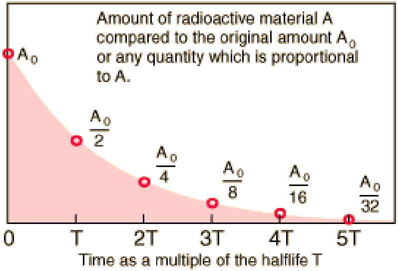

- Let $A(t)$ be the amount of a radioactive material present at time t and $k < 0$ the decay constant. Then

$$A'(t) = kA(t)$$

gives:

$$A(t) = A(0)e^{kt}.$$

- Let T_{half} denote the half-life. Then $\frac{1}{2}A(0) = A(0)e^{kT_{\text{half}}}$,

$$T_{\text{half}} = \frac{-\ln 2}{k} \approx \frac{-0.69}{k}.$$

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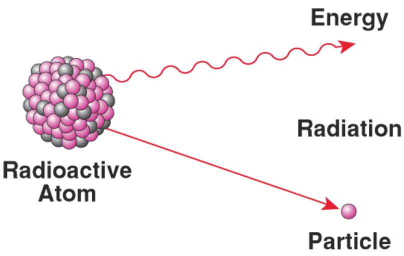

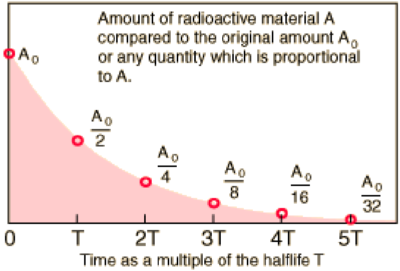

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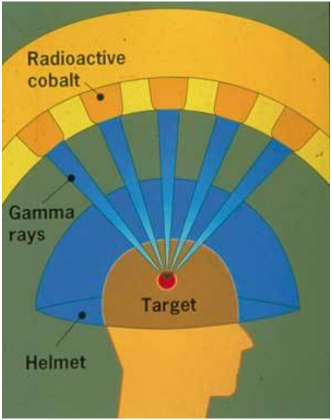
Cobalt-60



Gamma Knife Radiosurgery Unit (www1.wfubmc.edu)

- Cobalt-60, used extensively in medical radiology, has a half-life of 5.3 years. The decay constant k is given by

$$k = \frac{-\ln 2}{T_{\text{half}}} \approx \frac{-0.69}{5.3} \approx -0.131.$$

- If the initial sample of cobalt-60 has a mass of 100 grams, then the amount of the sample that will remain t years after is

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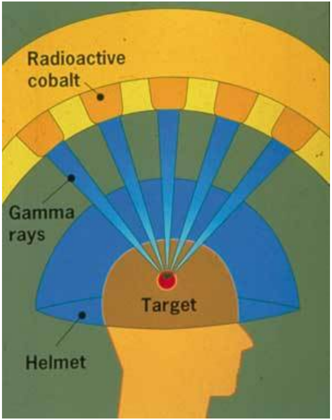

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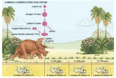
Radiocarbon Dating: Carbon-14


- The mathematics of radioactive decay is useful for many branches of science far removed from nuclear physics.
- One reason is that, in the late 1940's, Willard F. Libby discovered natural carbon-14 (radiocarbon), a radioactive isotope of carbon with a half-life of 5730 years.
- W. F. Libby was awarded the Nobel Prize in 1960 for his work on carbon-14 and its use in dating archaeological artifacts, and natural tritium, and its use in hydrology and geophysics.

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 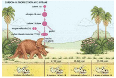   

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 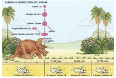   

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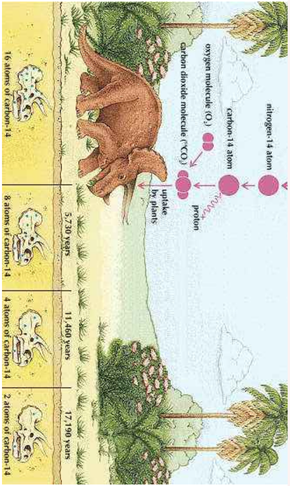

- All living organisms take in carbon through their food supply. While living, the ratio of radiocarbon to nonradioactive carbon that makes up the organism stays constant, since the organism takes in a constant supply of both in its food. After it dies, however, it no longer takes in either form of carbon.
- The ratio of radiocarbon to nonradioactive carbon then decreases with time as the radiocarbon decays away. The ratio decreases exponentially with time, so a 5600-year-old organic object has about half the radiocarbon/carbon ratio as a living organic object of the same type today.

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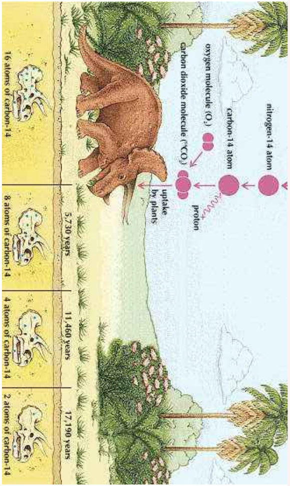

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 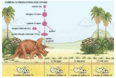   

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- The Chauvet Cave in southern France contains the oldest known cave paintings, based on radiocarbon dating (30,000 to 32,000 years ago).
- The technique of radiocarbon dating has been used to date objects as old as 50,000 years and has therefore been of enormous significance to archaeologists and anthropologists.

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 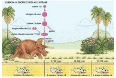   

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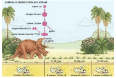


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Quiz (cont.)

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3. $\dfrac{d}{dx}e^{-x} =:$ (a) $-e^{-x}$, (b) e^{x}, (c) e^{-x}.

4. $\dfrac{d^2}{dx^2}e^{-x} =:$ (a) $-e^{-x}$, (b) e^{x}, (c) e^{-x}.

Compounding

The value, at the end of the year, of a principle of \$1000 invested at 6% compounded:

- annually (once per year):
  $A(1) = 1000(1 + 0.06) = \$1060.$
- quarterly (4 times per year):
  $A(1) = 1000(1 + (0.06/4))^4 \approx \$1061.36.$
- monthly (12 times per year):
  $A(1) = 1000(1+(0.06/12))^{12} \approx \$1061.67.$
- continuously:

$$A(1) = 1000 \lim_{n\to\infty} \left(1 + \frac{0.06}{n}\right)^n.$$

Let $x = \frac{n}{0.06}$.

$$A(1) = 1000 \left[\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x\right]^{0.06}$$

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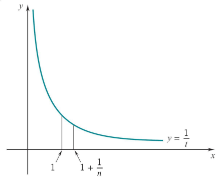

Theorem

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Proof.

- Define $g(h) = \begin{cases} \frac{1}{h}\ln(1+h), & h \in (-1,0)\cup(0,\infty), \\ 1 & h = 0 \end{cases}$
- At $x = 1$, the logarithm function has derivative

$$(\ln x)'|_{x=1} = \lim_{h\to 0}\frac{\ln(1+h)-\ln 1}{h} = \frac{1}{x}\Big|_{x=1}$$

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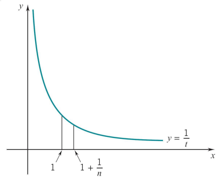

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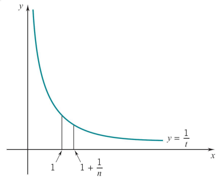

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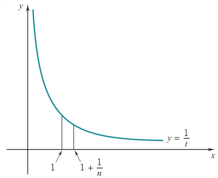

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Continuous Compound Interest

The economists' formula for continuous compounding is

$$A(t) = A(0)e^{rt}$$

- t is measured in years,
- $A(t)$ = the principle in dollars at time t,
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Outline

- Population Growth
  - Human Population Growth
- Radioactive Decay
  - Radioactive Decay
- Compound Interest
  - Compound Interest

Online Resources

- www.unesco.org/education
- www.populationinstitute.org
- www.globalchange.umich.edu
- www.geohive.com
- www.nrc.gov
- www.pollutionissues.com
- www1.wfubmc.edu/neurosurgery/Radiosurgery
- mathdl.maa.org
- www.britannica.com
- en.wikipedia.org