Lecture 7 Section 8.2 Integration by Parts

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1 Integration by Parts

1.1 Undoing the Product Rule

Integration by Parts: Undoing the Product Rule

Product Rule

- The *product rule*: $\frac{d}{dx}\big(uv\big) = u\frac{d}{dx}v + v\frac{d}{dx}u$
- In terms of differentials: $d\big(uv\big) = u\,dv + v\,du$
- Rearrange: $u\,dv = d\big(uv\big) - v\,du$
- Integrate: $\int u\,dv = uv - \int v\,du$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

- *Reduction*: poly. x^k can be reduced by diff.
- *Cycling*: $\sin x$, $\cos x$, e^x, $\sinh x$, $\cosh x$, $\cdots$, retain their form after diff. or int.
- *Change of Form*: $\ln x$, $\sin^{-1} x$, $\tan^{-1} x$, $\cdots$ completely change their form after diff.

Success depends on choosing u and dv so that $\int v\,du$ is *easier* that $\int u\,dv$

1.2 Reduction

Integration by Parts: Reduction
Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Reduction use the fact that poly. x^k can be reduced by diff.

Examples 1.

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x + C$$

$u = x,\ dv = \cos x\,dx,$ thus $du = dx,\ v = \sin x$ [2ex] $\int v\,du = \int \sin x\,dx$ is *easier* than $\int u\,dv = \int x\cos x\,dx$

$$\int xe^{-\frac{x}{2}}\,dx = x\big(-2e^{-\frac{x}{2}}\big) - \int \big(-2e^{-\frac{x}{2}}\big)\,dx$$
$$= -2xe^{-\frac{x}{2}} - 4e^{-\frac{x}{2}} + C$$

$u = x,\ dv = e^{-\frac{x}{2}}\,dx,$ thus $du = dx,\ v = -2e^{-\frac{x}{2}}$ [2ex] $\int v\,du = \int \big(-2e^{-\frac{x}{2}}\big)\,dx$ is *easier* than $\int u\,dv = \int xe^{-\frac{x}{2}}\,dx$

$$\int x^2\sin x\,dx = x^2\big(-\cos x\big) - \int \big(-\cos x\big)2x\,dx$$
$$= -x^2\cos x + 2\int x\cos x\,dx$$
$$= -x^2\cos x + 2\big(x\sin x + \cos x\big) + C$$

$u = x^2$, $dv = \sin x\,dx$, thus $du = 2xdx$, $v = -\cos x$ [2ex] $\int v\,du = \int(-\cos x)2x\,dx$ is *easier* than $\int u\,dv = \int x^2\sin x\,dx$

$$\int x^5\cos x^3\,dx = \frac{1}{3}x^3\sin x^3 - \int x^2\sin x^3\,dx$$
$$= \frac{1}{3}x^3\sin x^3 + \frac{1}{3}\cos x^3 + C$$

$u = x^3,\ dv = x^2\cos x^3\,dx,$ thus $du = 3x^2dx,\ v = \frac{1}{3}\sin x^3$ [2ex] $\int x^2\sin x^3\,dx = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos x^3 + C$

Reduction Formulas

$$\int u\,dv = uv - \int v\,du$$

If $F'(x) = f(x)$, then $$\int x^n f(x)\,dx = x^n F(x) - n\int x^{n-1}F(x)dx$$
(Set $u = x^n,\ dv = f(x)\,dx,\ du = nx^{n-1}\,dx,\ v = F(x)$)

- With $f(x) = e^{ax}$ and $F(x) = \frac{1}{a}e^{ax}$, $$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx$$
- With $f(x) = \cos(kx)$ and $F(x) = \frac{1}{k}\sin(kx)$, $$\int x^n \cos(kx)\,dx = \frac{1}{k}x^n \sin(kx) - \frac{n}{k}\int x^{n-1}\sin($$
- With $f(x) = \sin(kx)$ and $F(x) = -\frac{1}{k}\cos(kx)$, $$\int x^n \sin(kx)\,dx = -\frac{1}{k}x^n \cos(kx) + \frac{n}{k}\int x^{n-1}\text{co}$$

Reduction Formulas

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx.$$

Examples 2.

$$\begin{aligned}\int x^3 e^{2x}\,dx &= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\int x^2e^{2x}dx\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2e^{2x} - \int xe^{2x}dx\right)\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx\right)\right)\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right)\right) + C\\ &= \frac{1}{8}e^{2x}\left(4x^3 - 6x^2 + 6x - 3\right) + C\end{aligned}$$

1.3 Cycling

Integration by Parts: Cycling

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Cycling use the fact that functions such as $\sin x$, $\cos x$, e^x, $\sinh x$, $\cosh x$, $\cdots$, retain their form after differentiation or integration.

Example 3.

$$\begin{aligned}\int e^x\cos x\,dx &= e^x\sin x - \int e^x\sin x\,dx\\ &= e^x\sin x + e^x\cos x - \int e^x\cos x\,dx\end{aligned}$$

$u = e^x$, $dv = \cos x\,dx$, thus $du = e^x dx$, $v = \sin x$ [1ex] $\int e^x \sin x\,dx$ is of the *same form* as $\int e^x\cos x\,dx = e^x(-\cos x) - \int e^x(-\cos x)\,dx$ [1ex] $u = e^x$, $dv = \sin x\,dx$, thus $du = e^x dx$, $v = -\cos x$ $\int e^x\sin x\,dx$ is *cycling* after two integrations by parts. "Solve" the equation for $\int e^x\cos x\,dx$:

$$\begin{aligned}2\int e^x\cos x\,dx &= e^x\left(\sin x + \cos x\right) + C\\ \int e^x\cos x\,dx &= \frac{1}{2}e^x\left(\sin x + \cos x\right) + C\end{aligned}$$

1.4 Change of Form

Integration by Parts: Change of Form
Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Change of form use the fact that functions $\ln x$, $\sin^{-1} x$, $\tan^{-1} x$, $\cdots$ completely change their form after differentiation.

Examples 4.

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int x x^{-1}\,dx\\ &= x\ln x - \int dx = x\ln x - x + C\end{aligned}$$

$u = \ln x$, $dv = dx$, thus $du = x^{-1}dx$, $v = x$ [1ex] $\int v\,du = \int dx$ is *easier* than $\int u\,dv = \int \ln x\,dx$

$$\begin{aligned}\int \tan^{-1} x\,dx &= x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx\\ &= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + C\end{aligned}$$

$u = \tan^{-1} x$, $dv = dx$, thus $du = \frac{1}{1+x^2}\,dx$, $v = x$ [1ex] Set $u = 1+x^2$, $du = 2x\,dx$, $\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$$\begin{aligned}\int x^2\tan^{-1} x\,dx &= \frac{1}{3}x^3\tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx\\ &= \frac{1}{3}x^3\tan^{-1} x - \frac{1}{3}\int \big(x - \frac{x}{1+x^2}\big)\,dx\\ &= \frac{1}{3}x^3\tan^{-1} x - \frac{1}{6}\big(x^2 - \ln|1+x^2|\big) + C\end{aligned}$$

$u = \tan^{-1} x$, $dv = x^2\,dx$, thus $du = \frac{1}{1+x^2}\,dx$, $v = \frac{1}{3}x^3$ Note that $\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$

$$\begin{aligned}\int \cos^{-1} x\,dx &= x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}\,dx\\ &= x\cos^{-1} x - \sqrt{1-x^2} + C\end{aligned}$$

$u = \cos^{-1} x$, $dv = dx$, thus $du = -\frac{1}{\sqrt{1-x^2}}\,dx$, $v = x$ [1ex] Set $u = 1-x^2$, $du = -2x\,dx$, $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = -\sqrt{u} + C$

2 Parts on Definite Integrals

Parts on Definite Integrals
Parts on Definite Integrals

$$\int_a^b u\,dv = (uv)|_a^b - \int_a^b v\,du$$

Example 5.

$$\int_0^2 x^3\sqrt{4-x^2}\,dx = -\frac{1}{3}x^2(4-x^2)^{\frac{3}{2}}\bigg|_0^2 - \frac{1}{3}\int_0^2 (4-x^2)^{\frac{3}{2}}(-2x)\,dx$$
$$= -\frac{1}{3}\cdot\left(-\frac{64}{5}\right) = \frac{64}{15}$$

$u = x^2$, $dv = x\sqrt{4-x^2}\,dx$, thus $du = 2x\,dx$, $v = -\frac{1}{3}(4-x^2)^{\frac{3}{2}}$ [1ex] Set $u = 4-x^2$, $du = -2x\,dx$, $\displaystyle\int_0^2 (4-x^2)^{\frac{3}{2}}(-2x)\,dx = \int_4^0 u^{\frac{3}{2}}\,du = -\frac{2}{5}u^{\frac{5}{2}}\bigg|_0^4 = -\frac{64}{5}$.

Outline

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