Lecture 8 Section 8.3 Powers and Products of Trigonometric Functions

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1 Sine and Cosine

1.1 Odd Powers

$\int \sin^m x \cos x \, dx$**:** $u(= \sin x)$**-Substitution**

$$\int \sin^m x \cos x \, dx$$

Set $u = \sin x, \; du = \cos x \, dx$

- For $m \neq -1$,

$$\int \sin^m x \cos x \, dx = \int u^m \, du = \frac{1}{m+1} u^{m+1} + C$$
$$= \frac{1}{m+1} \sin^{m+1} x + C$$

- For $m = -1$,

$$\int \sin^{-1} x \cos x \, dx = \int u^{-1} \, du = \ln |u| + C$$
$$= \ln |\sin x| + C$$

$\int \sin x \cos^n x \, dx$**:** $u(= \cos x)$**-Substitution**

$$\int \sin x \cos^n x \, dx$$

Set $u = \cos x, \; du = -\sin x \, dx$

- For $n \neq -1$,

$$\int \sin x \cos^n x \, dx = -\int u^n \, du = -\frac{1}{n+1} u^{n+1} + C$$
$$= -\frac{1}{n+1} \cos^{n+1} x + C$$

- For $n = -1$,

$$\int \sin x \cos^{-1} x \, dx = -\int u^{-1} \, du = -\ln |u| + C$$
$$= -\ln |\cos x| + C$$

$\int \sin^m x \cos^n x\, dx$, $n > 1$ odd: : $u(= \sin x)$-Substitution

$$\int \sin^m x \cos^n x\, dx,\ n > 1 \text{ odd}$$

Set $u = \sin x$, $du = \cos x\, dx$. Note $\cos^2 x = 1 - \sin^2 x$

$$\begin{aligned}
\int \sin^m x \cos^n x\, dx &= \int \sin^m x \big(\cos^{n-1} x \cos x\big)\, dx \\
&= \int \sin^m x \big(\cos^2 x\big)^{\frac{n-1}{2}} \cos x\, dx \\
&= \int \sin^m x \big(1 - \sin^2 x\big)^{\frac{n-1}{2}}\, d\sin x \\
&= \int u^m \big(1 - u^2\big)^{\frac{n-1}{2}}\, du = \int \text{ polynomial}
\end{aligned}$$

Write $\cos^n x = \cos^{n-1} x \cos x$. Write $\cos^{n-1} x = \big(\cos^2 x\big)^{\frac{n-1}{2}}$. Use $\cos^2 x = 1 - \sin^2 x$. $u(= \sin x)$-Substitution

Example 1.

$$\begin{aligned}
\int \sin^2 x \cos^5 x\, dx &= \int u^2 \big(1 - u^2\big)^{\frac{5-1}{2}}\, du \\
&= \int (u^2 - 2u^4 + u^6)^2\, du \\
&= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C \\
&= \sin^3 x \left(\frac{1}{3} - \frac{2}{5}\sin^2 x + \frac{1}{7}\sin^4 x\right) + C
\end{aligned}$$

$\int \sin^m x \cos^n x\, dx$, $m > 1$ odd: : $u(= \cos x)$-Substitution

$$\int \sin^m x \cos^n x\, dx,\ m > 1 \text{ odd}$$

Set $u = \cos x$, $du = -\sin x\, dx$. Note $\sin^2 x = 1 - \cos^2 x$

$$\begin{aligned}
\int \sin^m x \cos^n x\, dx &= \int \big(\sin^{m-1} x \sin x\big) \cos^n x\, dx \\
&= \int \big(\sin^2 x\big)^{\frac{m-1}{2}} \cos^n x \sin x\, dx \\
&= -\int \big(1 - \cos^2 x\big)^{\frac{m-1}{2}} \cos^n x\, d\cos x \\
&= -\int \big(1 - u^2\big)^{\frac{m-1}{2}} u^n\, du = \int \text{ polynomial}
\end{aligned}$$

Write $\sin^m x = \sin^{m-1} x \cos x$. Write $\sin^{m-1} x = \big(\sin^2 x\big)^{\frac{m-1}{2}}$. Use $\sin^2 x = 1 - \cos^2 x$. $u(= \cos x)$-Substitution

Example 2.

$$
\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= -\int \left(1-u^2\right)^{\frac{3-1}{2}} u^2 \, du \\
&= \int \left(u^4 - u^2\right) du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
&= \cos^3 x \left(\frac{1}{5}\cos^2 x - \frac{1}{3}\right) + C
\end{aligned}
$$

1.2 Even Powers

Power Reduction

Power Reduction Formulas

$$\sin x \cos x = \frac{1}{2}\sin 2x, \;\; \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x, \;\; \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

Examples 3.

$$
\begin{aligned}
\int \cos^2 x \, dx &= \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx \\
&= \frac{1}{2}x + \frac{1}{4}\sin 2x + C \\
\int \cos^4 x \, dx &= \int \left(\cos^2\right)^2 x \, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^2 dx \\
&= \int \left(\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right) dx \\
&= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{4}\int \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx = \cdots
\end{aligned}
$$

$$
\begin{aligned}
\int \sin^2 x \, \cos^2 x \, dx &= \int \Big(\sin x \cos x\Big)^2 dx \\
&= \frac{1}{4}\int \sin^2 2x \, dx \\
&= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx \\
&= \frac{1}{8}x - \frac{1}{32}\sin 4x + C
\end{aligned}
$$

1.3 Powers of Sine or Cosine

Reduction by Parts

Reduction Formulas

$$
\begin{aligned}
\int \sin^n x \, dx &= -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx \\
\int \cos^n x \, dx &= \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx
\end{aligned}
$$

Proof.
Set $u = \sin^{n-1} x$, $dv = \sin x\, dx$, $du = (n-1)\sin^{n-2} x \cos x\, dx$, $v = -\cos x$.

$$\begin{aligned}
\int \sin^n x\, dx &= \int \sin^{n-1} \sin x, dx \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x\, dx \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x)\, dx \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x)\, dx \\
&\quad - (n-1) \int \sin^n x\, dx
\end{aligned}$$

Rearrange and solve the equation for $\int \sin^n x\, dx$.

Examples 4.

$$\begin{aligned}
\int \sin^2 x\, dx &= -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx \\
&= -\frac{1}{2} \sin x \cos x + \frac{1}{2} x + C \\
\int \sin^5 x\, dx &= -\frac{1}{5} \sin^4 x \cos x + \frac{4}{5} \int \sin^3 x\, dx \\
&= -\frac{1}{5} \sin^4 x \cos x + \frac{4}{5} \left(-\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x\, dx \right) \\
&= -\frac{1}{5} \sin^4 x \cos x - \frac{4}{15} \sin^2 x \cos x - \frac{8}{15} \cos x + C
\end{aligned}$$

1.4 Products of Sine and Cosine

Products of Sine and Cosine
Product-to-Sum Formulas

$$\begin{aligned}
\sin A \cos B &= \frac{1}{2} \left(\sin(A-B) + \sin(A+B) \right) \\
\sin A \sin B &= \frac{1}{2} \left(\cos(A-B) - \cos(A+B) \right) \\
\cos A \cos B &= \frac{1}{2} \left(\cos(A-B) + \cos(A+B) \right)
\end{aligned}$$

Example 5.

$$\begin{aligned}
\int \sin 5x \sin 2x\, dx &= \int \frac{1}{2} \left(\cos(5x - 2x) - \cos(5x + 2x) \right)\, dx \\
&= \frac{1}{2} \int \left(\cos 3x - \cos 7x \right)\, dx \\
&= \frac{1}{6} \sin 3x - \frac{1}{14} \sin 7x + C
\end{aligned}$$

1.5 Summary

Summary

Summary

$$\int \sin^m x \cos^n x \, dx$$

- Odd power of sine: substitute $u = \cos x$, $du = -\sin x \, dx$
- Odd power of cosine: substitute $u = \sin x$, $du = \cos x \, dx$
- Both powers even: Reduce powers $\sin x \cos x = \frac{1}{2} \sin 2x$, $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$, $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$ or integration by parts
- Remember the formulas

$$\cos^2 x + \sin^2 x = 1$$
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

Outline

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