Lecture 9

Section 8.4 Integrals Involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$ Trigonometric Substitutions

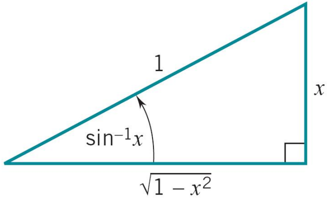
Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
http://math.uh.edu/∼jiwenhe/Math1432

$$\int \sqrt{a^2 - x^2}\, dx, \quad \int \sqrt{a^2 + x^2}\, dx, \quad \int \sqrt{x^2 - a^2}\, dx$$

Sine Substitution: $\sqrt{1-x^2}$



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$$1-\sin^2 u = \cos^2 u$$

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Example

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$$= \int \left(\frac{1}{2}+\frac{1}{2}\cos 2u\right)du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$$

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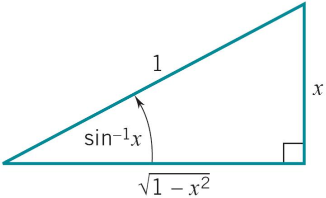

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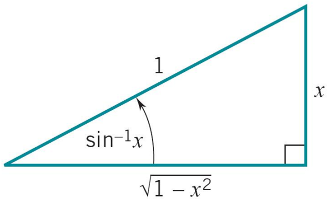

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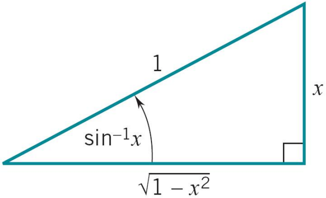

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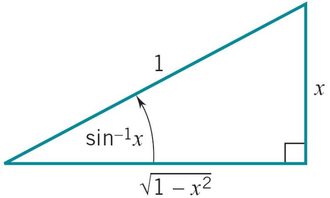

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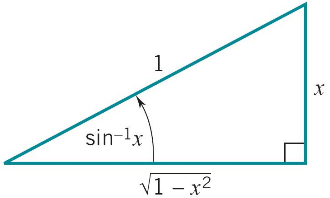

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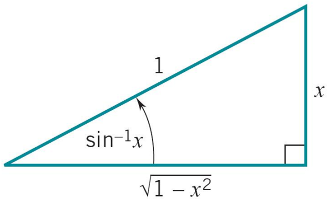

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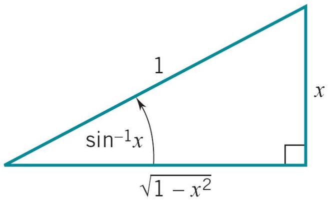

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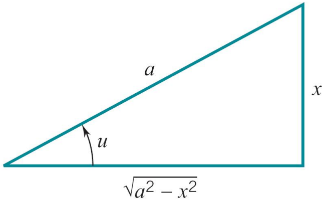

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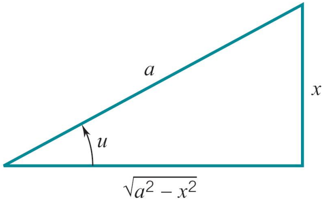

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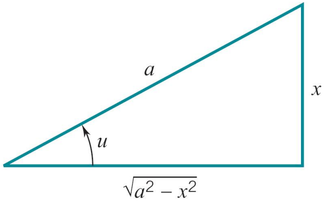

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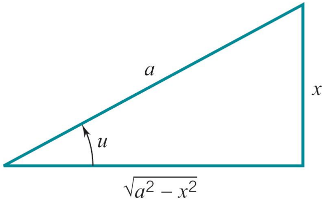

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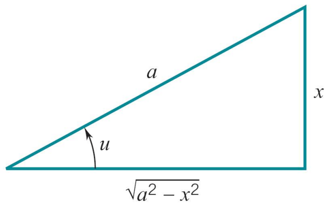

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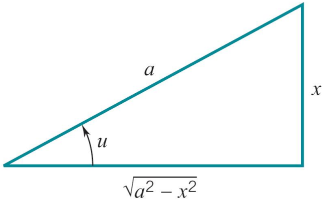

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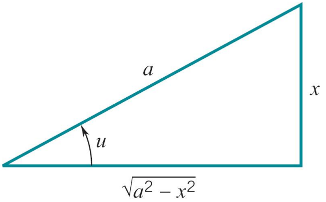

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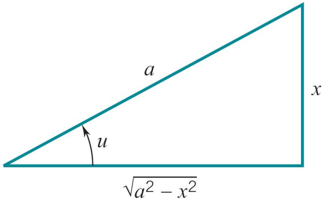

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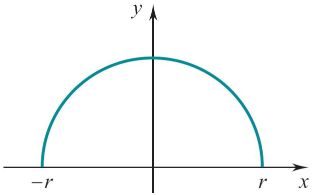

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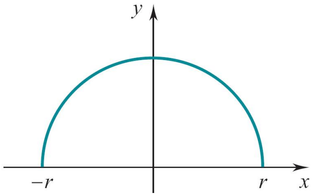
$=$ area enclosed by a semicircle of radius r

Example: Area of a Circle



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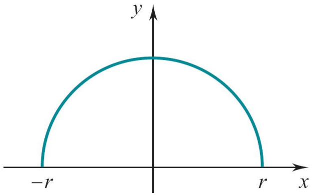

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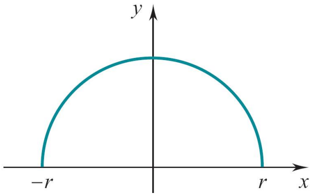

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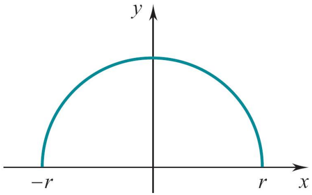

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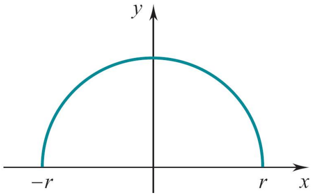

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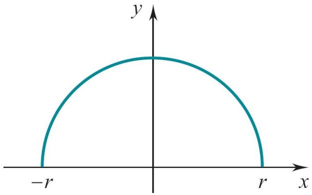
Example: Area of a Circle



Sine Substitution: $\sqrt{a^2 - x^2}$

$$\begin{aligned} 1 - \sin^2 u &= \cos^2 u \\ x &= a \sin u \\ dx &= a \cos u \, du \\ a^2 - x^2 &= a^2 \cos^2 u \end{aligned}$$

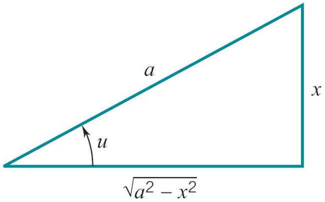
Example

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 \cos^2 u}\, r \cos u \, du = r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u \, du$$

$$= r^2 \left[\frac{1}{2} u + \frac{1}{4} \sin 2u \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi r^2}{2}$$

$=$ area enclosed by a semicircle of radius r

Example: Completing the Square



Sine Substitution: $\sqrt{a^2 - (x-c)^2}$

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Example

$$\int \sqrt{2 - x^2 + 4x}\, dx = \int \sqrt{2 + 4 - (x^2 + 4x + 4)}\, dx$$
$$= \int \sqrt{6 - (x-2)^2}\, dx = 6 \int \cos^2 u \, du = \frac{6}{2}(u + \sin u \cos u) + C$$
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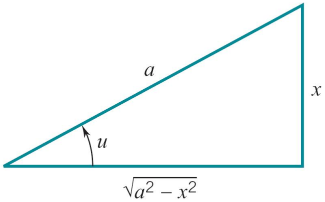

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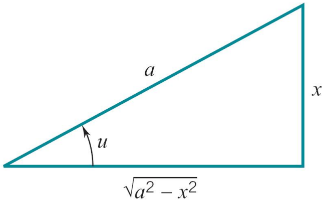

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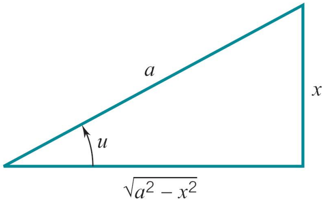

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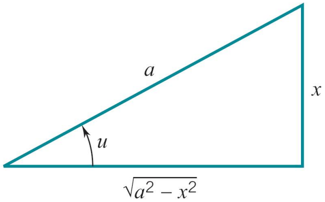

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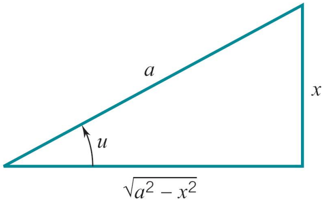

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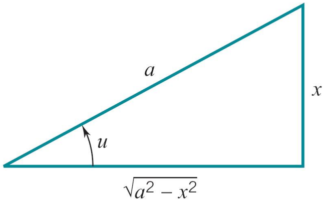

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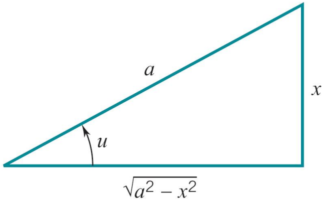

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Example

$$\int \frac{\sqrt{4 - x^2}}{x} dx = \int \frac{\sqrt{4 \cos^2 u}}{2 \sin u} 2 \cos u \, du = 2 \int \frac{\cos^2 u}{\sin u} du$$

$$= 2 \int \frac{1 - \sin^2 u}{\sin u} du = 2 \int \csc u - \sin u \, du$$

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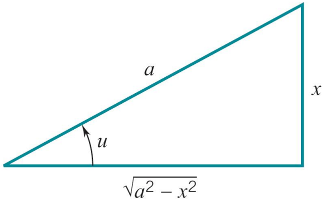

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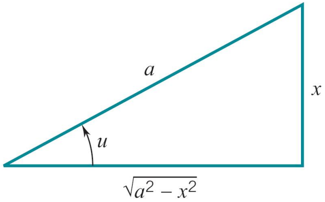

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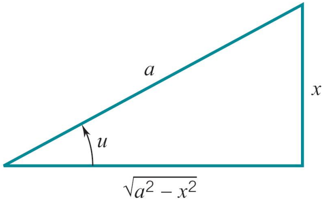

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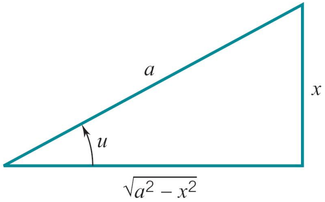

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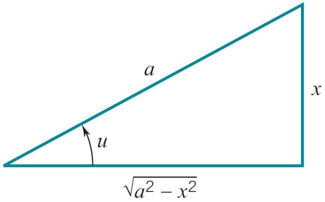

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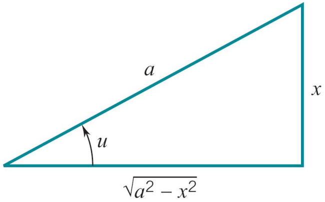

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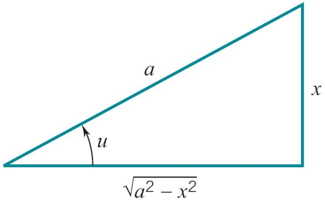

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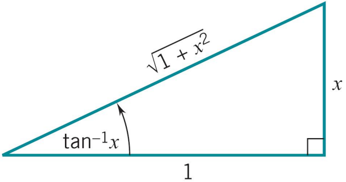
Tangent Substitution: $1 + x^2$



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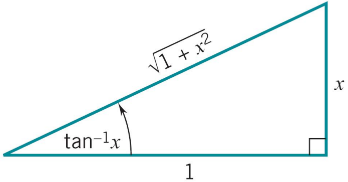

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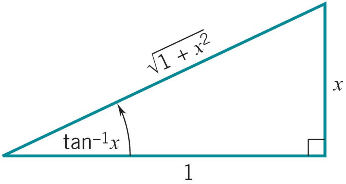

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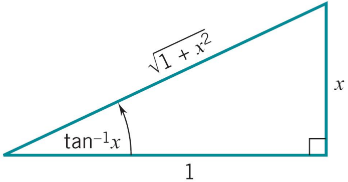

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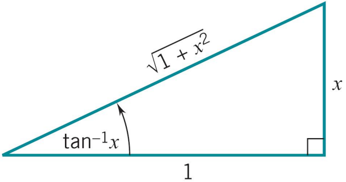

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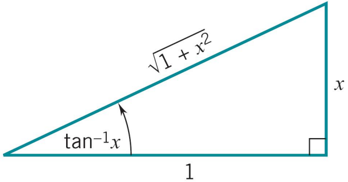
Tangent Substitution: $1 + x^2$



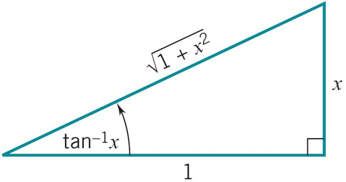
Tangent Substitution: $1 + x^2$

$$1 + \tan^2 u = \sec^2 u$$
$$x = \tan u$$
$$dx = \sec^2 u \, du$$
$$1 + x^2 = \sec^2 u$$

Example

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{\sec^4 u}\sec^2 u\,du = \int \cos^2 u\,du$$
$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 2u\right) du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$$
$$= \frac{1}{2}(u + \sin u \cos u) + C$$
$$= \frac{1}{2}\left(\tan^{-1} x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}\right) + C = \cdots$$

Tangent Substitution: $1 + x^2$



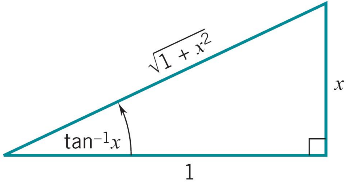
Tangent Substitution: $1 + x^2$

$$\begin{aligned} 1 + \tan^2 u &= \sec^2 u \\ x &= \tan u \\ dx &= \sec^2 u \, du \\ 1 + x^2 &= \sec^2 u \end{aligned}$$

Example

$$\begin{aligned} \int \frac{1}{(1+x^2)^2}\,dx &= \int \frac{1}{\sec^4 u} \sec^2 u \, du = \int \cos^2 u \, du \\ &= \int \left(\frac{1}{2} + \frac{1}{2} \cos 2u \right) du = \frac{1}{2} u + \frac{1}{4} \sin 2u + C \\ &= \frac{1}{2} (u + \sin u \cos u) + C \\ &= \frac{1}{2} \left(\tan^{-1} x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} \right) + C = \cdots \end{aligned}$$

Tangent Substitution: $1 + x^2$



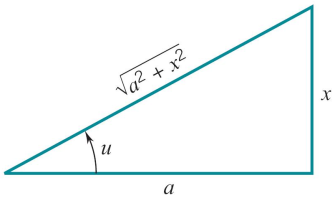
Tangent Substitution: $1 + x^2$

$$
\begin{aligned}
1 + \tan^2 u &= \sec^2 u \\
x &= \tan u \\
dx &= \sec^2 u \, du \\
1 + x^2 &= \sec^2 u
\end{aligned}
$$

Example

$$
\begin{aligned}
\int \frac{1}{(1+x^2)^2}\, dx &= \int \frac{1}{\sec^4 u} \sec^2 u \, du = \int \cos^2 u \, du \\
&= \int \left(\frac{1}{2} + \frac{1}{2} \cos 2u \right) du = \frac{1}{2} u + \frac{1}{4} \sin 2u + C \\
&= \frac{1}{2} (u + \sin u \cos u) + C \\
&= \frac{1}{2} \left(\tan^{-1} x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} \right) + C = \cdots
\end{aligned}
$$

Tangent Substitution: $\sqrt{a^2 + x^2}$



Tangent Substitution: $\sqrt{a^2 + x^2}$

$$\begin{aligned} 1 + \tan^2 u &= \sec^2 u \\ x &= a \tan u \\ dx &= a \sec^2 u \, du \\ a^2 + x^2 &= a^2 \sec^2 u \end{aligned}$$

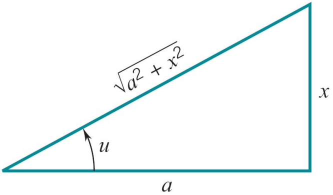
Example

$$\int \sqrt{a^2 + x^2}\, dx = \int \sqrt{a^2 \sec^2 u}\, a \sec^2 u \, du = a^2 \int \sec^3 u \, du$$

$$= \frac{a^2}{2}\left(\sec u \tan u + \ln|\sec u + \tan u|\right) + C$$

$$= \frac{a^2}{2}\left(\frac{\sqrt{a^2+x^2}}{a}\frac{x}{a} + \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|\right) + C = \cdots$$

Tangent Substitution: $\sqrt{a^2 + x^2}$



Tangent Substitution: $\sqrt{a^2 + x^2}$

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$$x = a \tan u$$
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Example

$$\int \sqrt{a^2 + x^2}\, dx = \int \sqrt{a^2 \sec^2 u}\, a \sec^2 u \, du = a^2 \int \sec^3 u \, du$$
$$= \frac{a^2}{2}\left(\sec u \tan u + \ln|\sec u + \tan u|\right) + C$$
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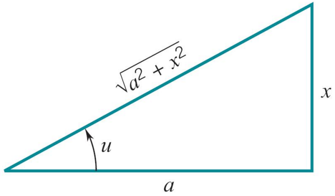

Tangent Substitution: $\sqrt{a^2 + x^2}$

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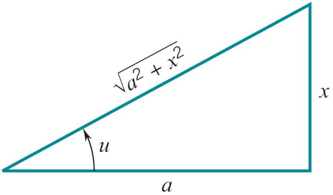
Example

$$\int \sqrt{a^2 + x^2}\, dx = \int \sqrt{a^2 \sec^2 u}\, a \sec^2 u \, du = a^2 \int \sec^3 u \, du$$

$$= \frac{a^2}{2} \left(\sec u \tan u + \ln |\sec u + \tan u| \right) + C$$

$$= \frac{a^2}{2} \left(\frac{\sqrt{a^2 + x^2}}{a} \frac{x}{a} + \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| \right) + C = \cdots$$

Tangent Substitution: $\sqrt{a^2+x^2}$



Tangent Substitution: $\sqrt{a^2+x^2}$

$$\begin{aligned} 1+\tan^2 u &= \sec^2 u \\ x &= a\tan u \\ dx &= a\sec^2 u\,du \\ a^2+x^2 &= a^2\sec^2 u \end{aligned}$$

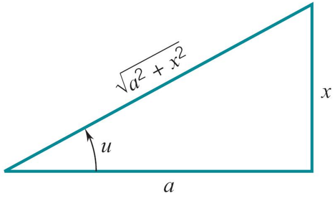
Example

$$\int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2\sec^2 u}\,a\sec^2 u\,du = a^2\int \sec^3 u\,du$$

$$= \frac{a^2}{2}\left(\sec u\tan u + \ln|\sec u+\tan u|\right) + C$$

$$= \frac{a^2}{2}\left(\frac{\sqrt{a^2+x^2}}{a}\frac{x}{a} + \ln\left|\frac{\sqrt{a^2+x^2}}{a}+\frac{x}{a}\right|\right) + C = \cdots$$

Tangent Substitution: $\sqrt{a^2 + x^2}$



Tangent Substitution: $\sqrt{a^2 + x^2}$

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\begin{aligned}
1 + \tan^2 u &= \sec^2 u \\
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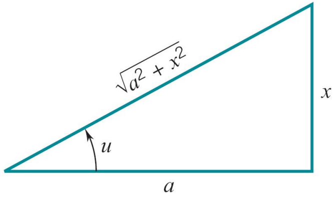
dx &= a \sec^2 u \, du \\
a^2 + x^2 &= a^2 \sec^2 u
\end{aligned}
$$

Example

$$
\begin{aligned}
\int \sqrt{a^2 + x^2}\, dx &= \int \sqrt{a^2 \sec^2 u}\, a \sec^2 u \, du = a^2 \int \sec^3 u \, du \\
&= \frac{a^2}{2} \left(\sec u \tan u + \ln |\sec u + \tan u| \right) + C \\
&= \frac{a^2}{2} \left(\frac{\sqrt{a^2 + x^2}}{a} \frac{x}{a} + \ln \left| \frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right| \right) + C = \cdots
\end{aligned}
$$

Tangent Substitution: $\sqrt{a^2+x^2}$



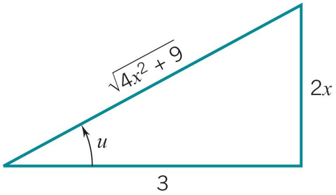
Tangent Substitution: $\sqrt{a^2+x^2}$

$$\begin{aligned} 1+\tan^2 u &= \sec^2 u \\ x &= a\tan u \\ dx &= a\sec^2 u\, du \\ a^2+x^2 &= a^2\sec^2 u \end{aligned}$$

Example

$$\begin{aligned} \int \sqrt{a^2+x^2}\,dx &= \int \sqrt{a^2\sec^2 u}\, a\sec^2 u\, du = a^2 \int \sec^3 u\, du \\ &= \frac{a^2}{2}\left(\sec u\tan u + \ln\left|\sec u + \tan u\right|\right) + C \\ &= \frac{a^2}{2}\left(\frac{\sqrt{a^2+x^2}}{a}\frac{x}{a} + \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|\right) + C = \cdots \end{aligned}$$

Example



Tangent Substitution: $\sqrt{a^2 + x^2}$

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1 + \tan^2 u &= \sec^2 u \\
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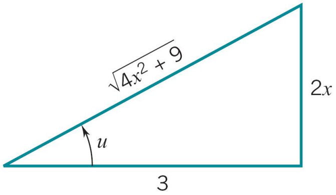
$$

Example

$$
\begin{aligned}
\int \frac{1}{x\sqrt{9+4x^2}}\,dx &= \frac{1}{2}\int \frac{1}{\frac{3}{2}\tan u \sqrt{\frac{9}{4}\sec^2 u}}\,\frac{3}{2}\sec^2 u\,du \\
&= \frac{1}{3}\int \frac{\sec u}{\tan u}\,du = \frac{1}{3}\int \csc u\,du \\
&= \frac{1}{3}\ln|\csc u - \cot u| + C \\
&= \frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}-3}{2x}\right| + C
\end{aligned}
$$

Example



Tangent Substitution: $\sqrt{a^2+x^2}$

$$1+\tan^2 u = \sec^2 u$$
$$x = a\tan u$$
$$dx = a\sec^2 u\, du$$
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Example

$$\int \frac{1}{x\sqrt{9+4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\frac{3}{2}\tan u\sqrt{\frac{9}{4}\sec^2 u}}\,\frac{3}{2}\sec^2 u\, du$$
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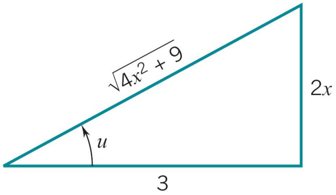
Example



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Example

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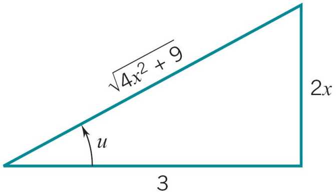
Example



Tangent Substitution: $\sqrt{a^2+x^2}$

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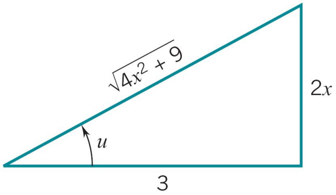
Example

$$
\begin{aligned}
\int \frac{1}{x\sqrt{9+4x^2}}\,dx &= \frac{1}{2}\int \frac{1}{\frac{3}{2}\tan u\sqrt{\frac{9}{4}\sec^2 u}}\,\frac{3}{2}\sec^2 u\,du\\
&= \frac{1}{3}\int \frac{\sec u}{\tan u}\,du = \frac{1}{3}\int \csc u\,du\\
&= \frac{1}{3}\ln|\csc u-\cot u|+C\\
&= \frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}-3}{2x}\right|+C
\end{aligned}
$$

Example



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Example

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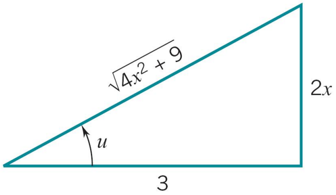
Example



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Example

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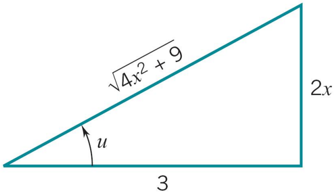
Example



Tangent Substitution: $\sqrt{a^2+x^2}$

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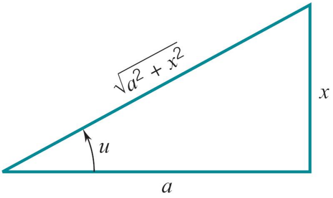
Example

$$\begin{aligned} \int \frac{1}{x\sqrt{9+4x^2}}\,dx &= \frac{1}{2}\int \frac{1}{\frac{3}{2}\tan u\sqrt{\frac{9}{4}\sec^2 u}}\,\frac{3}{2}\sec^2 u\, du \\ &= \frac{1}{3}\int \frac{\sec u}{\tan u}\,du = \frac{1}{3}\int \csc u\, du \\ &= \frac{1}{3}\ln|\csc u - \cot u| + C \\ &= \frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}-3}{2x}\right| + C \end{aligned}$$

Example



Tangent Substitution: $\sqrt{a^2+x^2}$

$$\begin{aligned} 1+\tan^2 u &= \sec^2 u \\ x &= a\tan u \\ dx &= a\sec^2 u\,du \\ a^2+x^2 &= a^2\sec^2 u \end{aligned}$$

Example

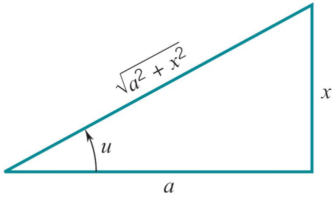
$$\int \frac{x^2}{\sqrt{4+x^2}}\,dx = \int \frac{4\tan^2 u}{\sqrt{4+4\tan^2 u}} 2\sec^2 u\,du = \int \frac{8\tan^2 u\sec^2 u}{\sqrt{4\sec^2 u}}\,du$$

$$= \int \frac{8\tan^2 u\sec^2 u}{2\sec u}\,du = 4\int \tan^2 u\sec u\,du$$

finish as in Example 12, Section 8.3

substitute in terms of x

Example



Tangent Substitution: $\sqrt{a^2 + x^2}$

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Example

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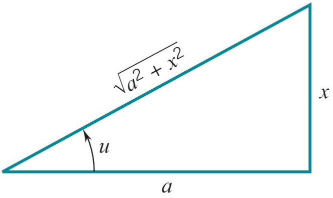
substitute in terms of x

Example



Tangent Substitution: $\sqrt{a^2+x^2}$

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\begin{aligned}
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Example

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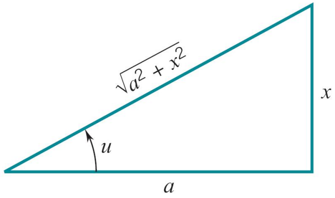
finish as in Example 12, Section 8.3

substitute in terms of x

Example



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Example

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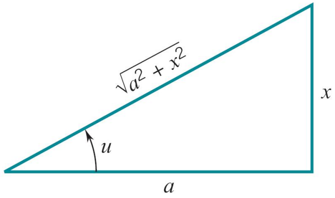
finish as in Example 12, Section 8.3

substitute in terms of x

Example



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Example

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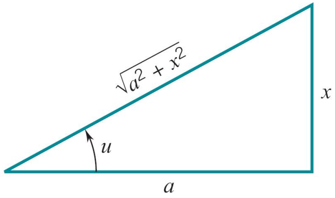
finish as in Example 12, Section 8.3

substitute in terms of x

Example



Tangent Substitution: $\sqrt{a^2+x^2}$

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Example

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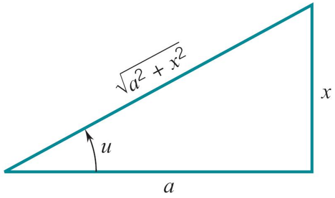
finish as in Example 12, Section 8.3

substitute in terms of x

Example



Tangent Substitution: $\sqrt{a^2+x^2}$

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$$x = a\tan u$$
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$$a^2+x^2 = a^2\sec^2 u$$

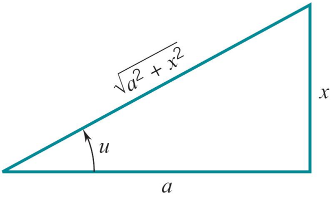
Example

$$\int \frac{x^2}{\sqrt{4+x^2}}\,dx = \int \frac{4\tan^2 u}{\sqrt{4+4\tan^2 u}} 2\sec^2 u\,du = \int \frac{8\tan^2 u \sec^2 u}{\sqrt{4\sec^2 u}}\,du$$
$$= \int \frac{8\tan^2 u\sec^2 u}{2\sec u}\,du = 4\int \tan^2 u \sec u\,du$$

finish as in Example 12, Section 8.3

substitute in terms of x

Example: Completing the Square



Tangent Substitution: $\sqrt{a^2+(x-c)^2}$

$$1+\tan^2 u = \sec^2 u$$
$$x - c = a\tan u$$
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Example

$$\int x\sqrt{16+x^2-6x}\,dx = \int x\sqrt{16-9+(x^2-6x+9)}\,dx$$
$$= \int x\sqrt{7+(x-3)^2}\,dx$$
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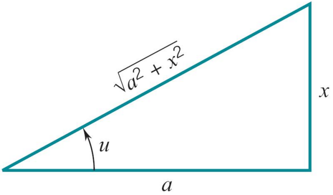
Finish it!

Example: Completing the Square



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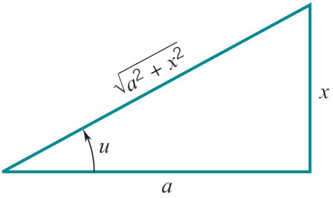
Finish it!

Example: Completing the Square



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Finish it!

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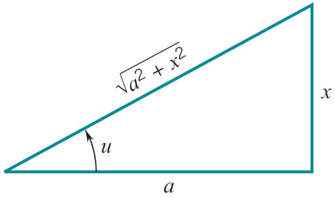

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Finish it!

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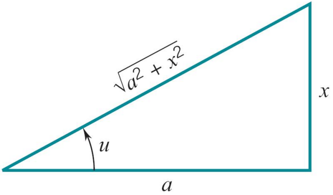

Tangent Substitution: $\sqrt{a^2+(x-c)^2}$

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x - c &= a \tan u \\
dx &= a \sec^2 u \, du \\
a^2 + (x-c)^2 &= a^2 \sec^2 u
\end{aligned}
$$

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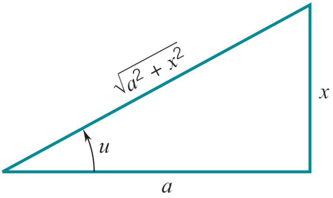

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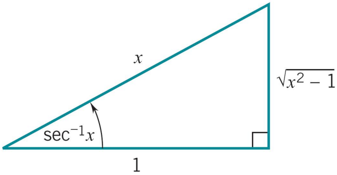
Finish it!

Secant Substitution: $x^2 - 1$



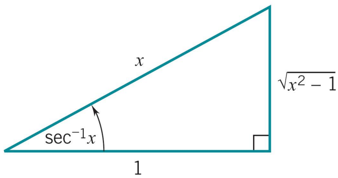
Secant Substitution: $x^2 - 1$

$$\sec^2 u - 1 = \tan^2 u$$
$$x = \sec u$$
$$dx = \sec u \tan u \, du$$
$$x^2 - 1 = \tan^2 u$$

Example

$$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int \frac{\sec u \tan u}{\sec^2 u \sqrt{\tan^2 u}}\,du = \int \frac{1}{\sec u}\,du$$
$$= \int \cos u \, du = \sin u + C$$
$$= \frac{\sqrt{x^2-1}}{x} + C$$

Secant Substitution: $x^2 - 1$



Secant Substitution: $x^2 - 1$

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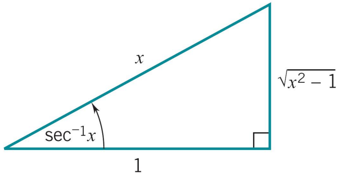

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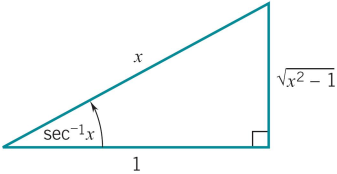

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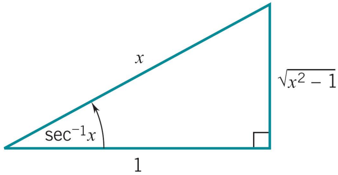

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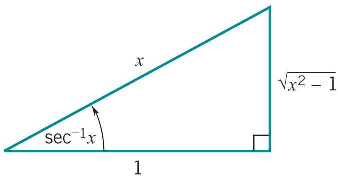

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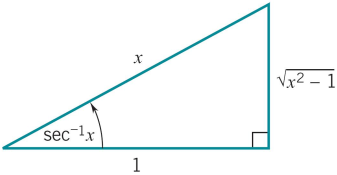

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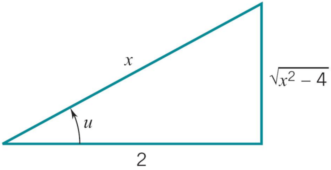
Secant Substitution: $\sqrt{x^2 - a^2}$



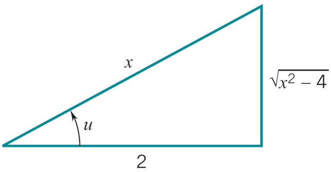
Secant Substitution: $\sqrt{x^2 - a^2}$

$$\begin{aligned} \sec^2 u - 1 &= \tan^2 u \\ x &= a \sec u \\ dx &= a \sec u \tan u \, du \\ x^2 - a^2 &= a^2 \tan^2 u \end{aligned}$$

Example

$$\begin{aligned} \int \frac{1}{x(x^2-4)^{3/2}} \, dx &= \int \frac{2 \sec u \tan u}{2 \sec u (4 \tan^2 u)^{3/2}} \, du = \frac{1}{8} \int \frac{1}{\tan^2 u} \, du \\ &= \frac{1}{8} \int \cot^2 u \, du = \frac{1}{8} \int (\csc^2 u - 1) \, du \\ &= \frac{1}{8} (-\cot u - u) + C \\ &= -\frac{1}{8} \left(\frac{2}{\sqrt{x^2-4}} + \sec^{-1}(x/2) \right) + C \end{aligned}$$

Secant Substitution: $\sqrt{x^2 - a^2}$



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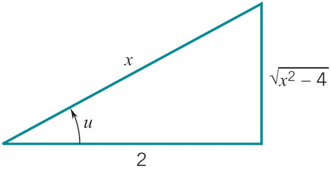

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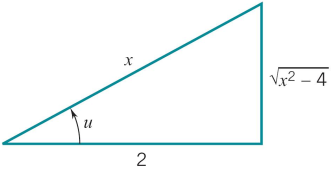

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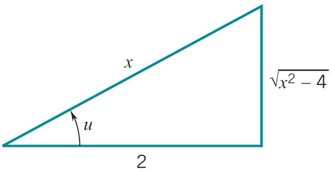

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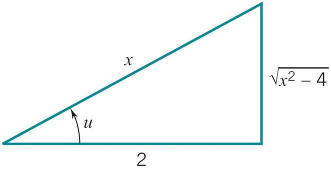

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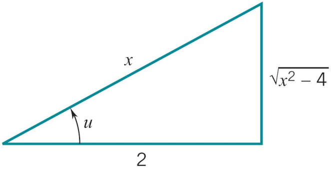

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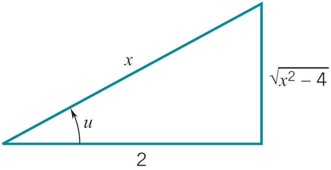

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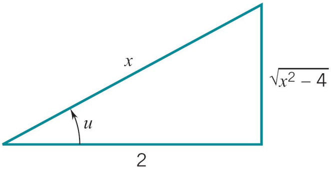
Example: Completing the Square



Secant Substitution: $\sqrt{x^2 - a^2}$

$$\sec^2 u - 1 = \tan^2 u$$
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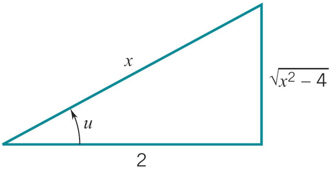
$$dx = a \sec u \tan u \, du$$
$$(x - c)^2 - a^2 = a^2 \tan^2 u$$

Example

$$\int \frac{x}{\sqrt{x^2 + 2x - 3}} \, dx = \int \frac{x}{\sqrt{(x^2 + 2x + 1) - 316 - 1}} \, dx$$
$$= \int \frac{x}{\sqrt{(x+1)^2 - 4}}, dx$$
$$= \int \frac{2 \sec u - 1}{\sqrt{4 \tan^2 u}} 2 \sec u \tan u \, du$$

Finish it!

Example: Completing the Square



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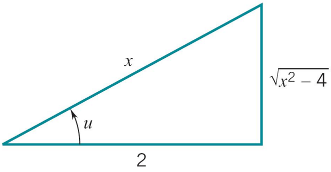
Finish it!

Example: Completing the Square



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Example

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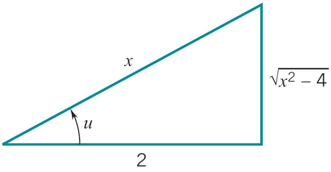
Finish it!

Example: Completing the Square



Secant Substitution: $\sqrt{x^2 - a^2}$

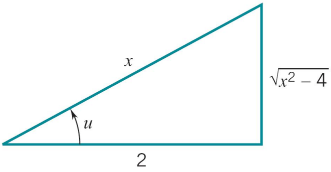
$$\sec^2 u - 1 = \tan^2 u$$
$$x - c = a \sec u$$
$$dx = a \sec u \tan u \, du$$
$$(x-c)^2 - a^2 = a^2 \tan^2 u$$

Example

$$\int \frac{x}{\sqrt{x^2+2x-3}}\,dx = \int \frac{x}{\sqrt{(x^2+2x+1) - 316 - 1}}\,dx$$
$$= \int \frac{x}{\sqrt{(x+1)^2 - 4}},dx$$
$$= \int \frac{2\sec u - 1}{\sqrt{4\tan^2 u}} 2\sec u \tan u \, du$$

Finish it!

Example: Completing the Square



Secant Substitution: $\sqrt{x^2 - a^2}$

$$
\begin{aligned}
\sec^2 u - 1 &= \tan^2 u \\
x - c &= a \sec u \\
dx &= a \sec u \tan u \, du \\
(x-c)^2 - a^2 &= a^2 \tan^2 u
\end{aligned}
$$

Example

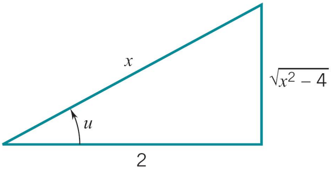
$$
\int \frac{x}{\sqrt{x^2+2x-3}}\, dx = \int \frac{x}{\sqrt{(x^2+2x+1) - 316 - 1}}\, dx
$$

$$
= \int \frac{x}{\sqrt{(x+1)^2 - 4}}, dx
$$

$$
= \int \frac{2\sec u - 1}{\sqrt{4\tan^2 u}} 2 \sec u \tan u \, du
$$

Finish it!

Example: Completing the Square



Secant Substitution: $\sqrt{x^2 - a^2}$

$$\begin{aligned} \sec^2 u - 1 &= \tan^2 u \\ x - c &= a \sec u \\ dx &= a \sec u \tan u \, du \\ (x-c)^2 - a^2 &= a^2 \tan^2 u \end{aligned}$$

Example

$$\int \frac{x}{\sqrt{x^2+2x-3}}\,dx = \int \frac{x}{\sqrt{(x^2+2x+1)-316-1}}\,dx$$

$$= \int \frac{x}{\sqrt{(x+1)^2-4}},\,dx$$

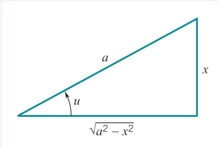
$$= \int \frac{2\sec u - 1}{\sqrt{4\tan^2 u}} 2\sec u \tan u \, du$$

Finish it!

Summary

$a^2 - x^2$

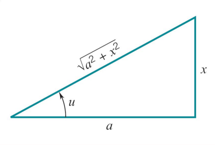
$$1 - \sin^2 u = \cos^2 u$$
$$x = a \sin u$$
$$dx = a \cos u \, du$$
$$a^2 - x^2 = a^2 \cos^2 u$$



$$\sin u = \frac{x}{a}$$
$$\cos u = \frac{\sqrt{a^2 - x^2}}{a}$$
$$\tan u = \frac{x}{\sqrt{a^2 - x^2}}$$

$a^2 + x^2$

$$1 + \tan^2 u = \sec^2 u$$
$$x = a \tan u$$
$$dx = a \sec^2 u \, du$$
$$a^2 + x^2 = a^2 \sec^2 u$$



$$\sin u = \frac{x}{\sqrt{a^2 + x^2}}$$
$$\cos u = \frac{a}{\sqrt{a^2 + x^2}}$$
$$\tan u = \frac{x}{a}$$

$x^2 - a^2$

$$\sec^2 u - 1 = \tan^2 u$$
$$x = a \sec u$$
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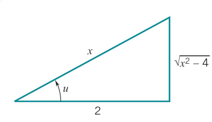
$$x^2 - a^2 = a^2 \tan^2 u$$



$$\sin u = \frac{\sqrt{x^2 - a^2}}{x}$$
$$\cos u = \frac{a}{x}$$
$$\tan u = \frac{\sqrt{x^2 - a^2}}{a}$$

Summary

$a^2 - x^2$

$$1 - \sin^2 u = \cos^2 u$$
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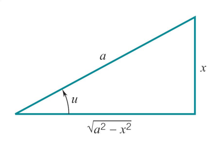

$$\sin u = \frac{x}{a}$$
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$a^2 + x^2$

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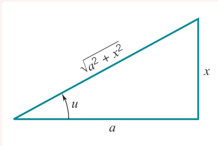

$$\sin u = \frac{x}{\sqrt{a^2 + x^2}}$$
$$\cos u = \frac{a}{\sqrt{a^2 + x^2}}$$
$$\tan u = \frac{x}{a}$$

$x^2 - a^2$

$$\sec^2 u - 1 = \tan^2 u$$
$$x = a \sec u$$
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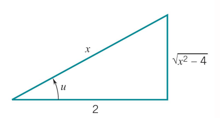

$$\sin u = \frac{\sqrt{x^2 - a^2}}{x}$$
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Summary

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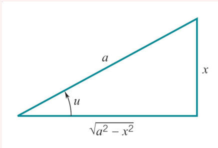

$$\sin u = \frac{x}{a}$$
$$\cos u = \frac{\sqrt{a^2 - x^2}}{a}$$
$$\tan u = \frac{x}{\sqrt{a^2 - x^2}}$$

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$$1 + \tan^2 u = \sec^2 u$$
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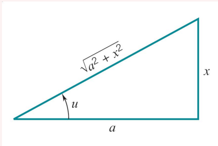

$$\sin u = \frac{x}{\sqrt{a^2 + x^2}}$$
$$\cos u = \frac{a}{\sqrt{a^2 + x^2}}$$
$$\tan u = \frac{x}{a}$$

$x^2 - a^2$

$$\sec^2 u - 1 = \tan^2 u$$
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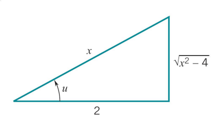
$$x^2 - a^2 = a^2 \tan^2 u$$



$$\sin u = \frac{\sqrt{x^2 - a^2}}{x}$$
$$\cos u = \frac{a}{x}$$
$$\tan u = \frac{\sqrt{x^2 - a^2}}{a}$$

Outline

- Trigonometric Substitution
  - Sine Substitution
  - Tangent Substitution
  - Secant Substitution