Lecture 9 Section 8.4 Integrals Involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$ Trigonometric Substitutions

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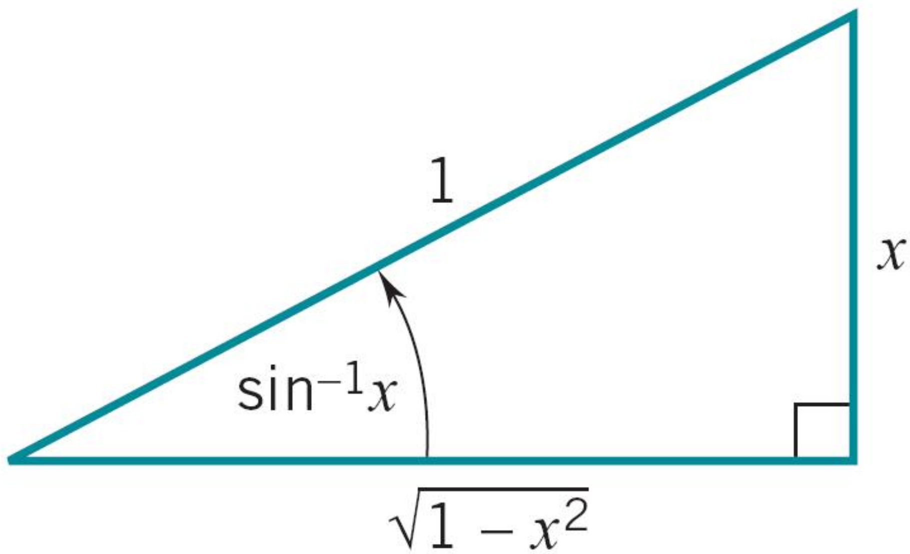
1 Trigonometric Substitution

1.1 Sine Substitution

Sine Substitution: $\sqrt{1 - x^2}$



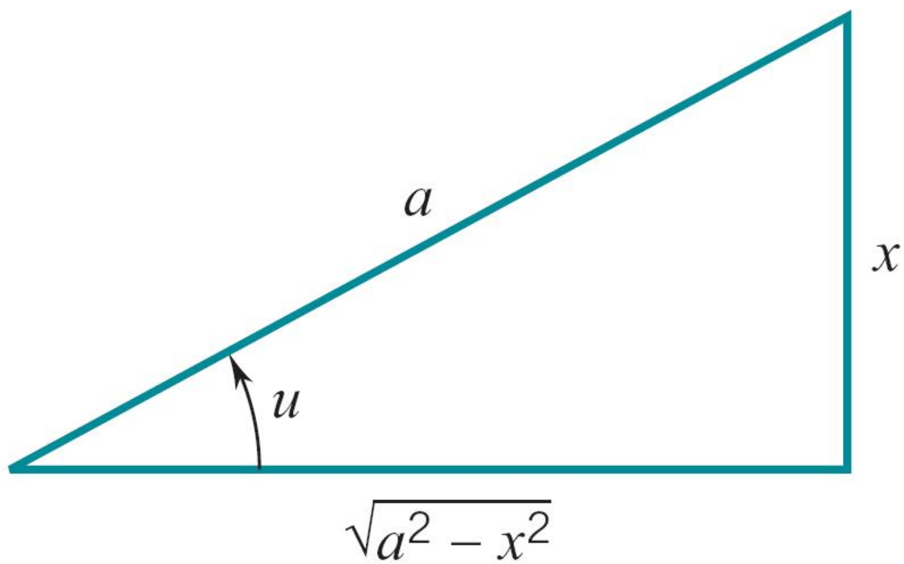
Sine Substitution: $\sqrt{1 - x^2}$

$$\begin{aligned} 1 - \sin^2 u &= \cos^2 u \\ x &= \sin u \\ dx &= \cos u \, du \\ 1 - x^2 &= \cos^2 u \end{aligned}$$

Example 1.

$$\begin{aligned}\int \sqrt{1-x^2}\,dx &= \int \sqrt{\cos^2 u}\cos u\,du = \int \cos^2 u\,du\\ &= \int \left(\frac{1}{2}+\frac{1}{2}\cos 2u\right)\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C\\ &= \frac{1}{2}\left(u+\sin u\cos u\right)+C\\ &= \frac{1}{2}\left(\sin^{-1}x + x\sqrt{1-x^2}\right)+C\end{aligned}$$

Sine Substitution: $\sqrt{a^2-x^2}$



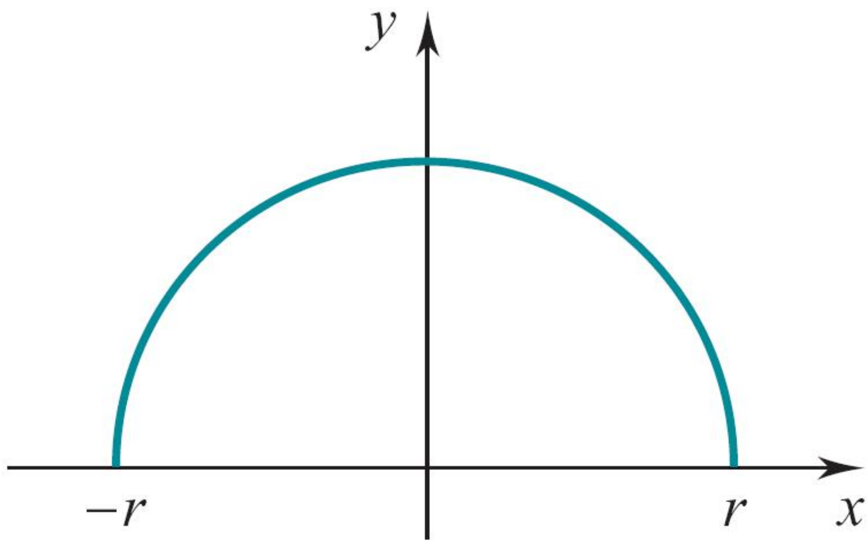
Sine Substitution: $\sqrt{a^2-x^2}$

$$\begin{aligned}1-\sin^2 u &= \cos^2 u\\ x &= a\sin u\\ dx &= a\cos u\,du\\ a^2-x^2 &= a^2\cos^2 u\end{aligned}$$

Example 2.

$$\begin{aligned}\int \sqrt{a^2 - x^2}\, dx &= \int \sqrt{a^2 \cos^2 u}\, a \cos u\, du = a^2 \int \cos^2 u\, du \\ &= a^2 \int \left(\frac{1}{2} + \frac{1}{2}\cos 2u\right)\, du = \frac{a^2}{2} u + \frac{a^2}{4} \sin 2u + C \\ &= \frac{a^2}{2}\left(u + \sin u \cos u\right) + C \\ &= \frac{a^2}{2}\left(\sin^{-1} x + x\sqrt{1 - x^2}\right) + C\end{aligned}$$

Example: Area of a Circle



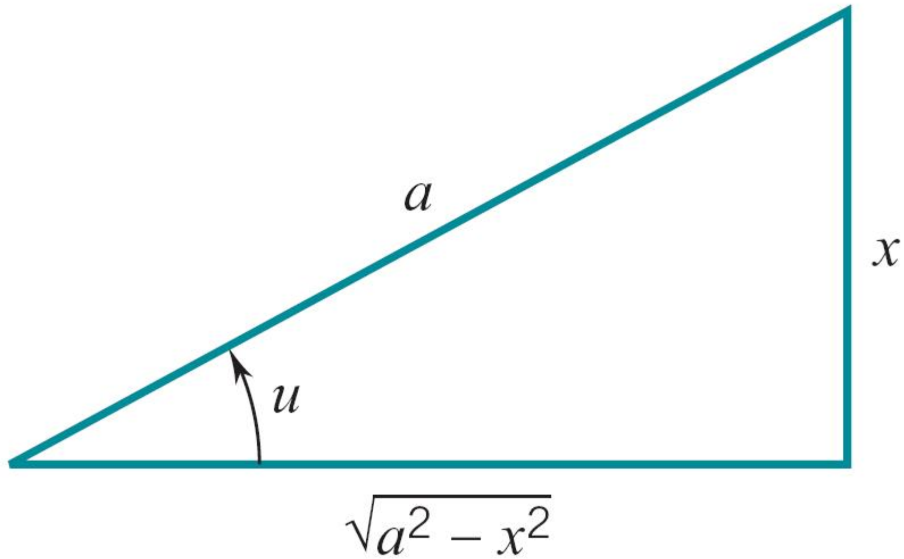
Sine Substitution: $\sqrt{a^2 - x^2}$

$$\begin{aligned}1 - \sin^2 u &= \cos^2 u \\ x &= a \sin u \\ dx &= a \cos u\, du \\ a^2 - x^2 &= a^2 \cos^2 u\end{aligned}$$

Example 3.

$$\begin{aligned}\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 \cos^2 u}\, r \cos u\, du = r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u\, du \\ &= r^2 \left[\frac{1}{2} u + \frac{1}{4} \sin 2u\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi r^2}{2} \\ &= \text{area enclosed by a semicircle of radius } r\end{aligned}$$

Example: Completing the Square



Sine Substitution: $\sqrt{a^2 - (x-c)^2}$

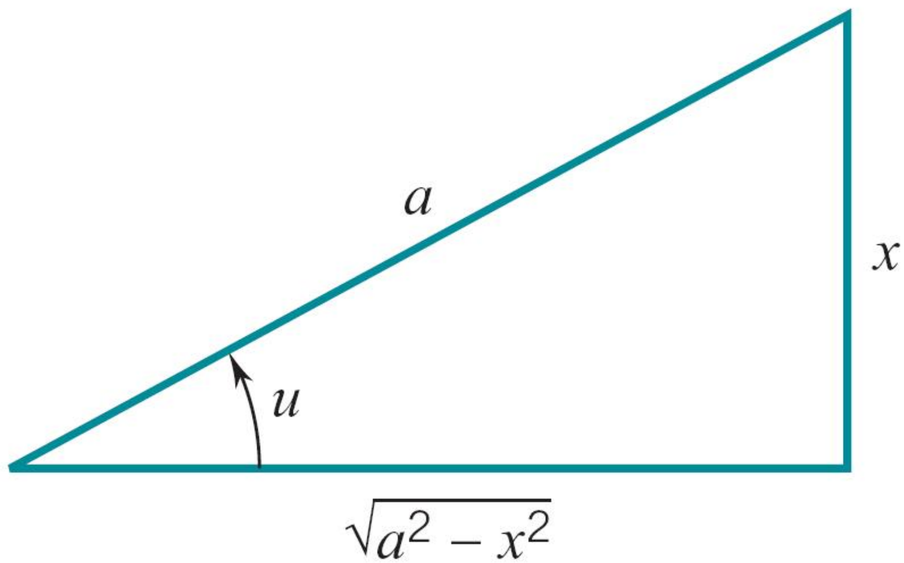
$$\begin{aligned} 1 - \sin^2 u &= \cos^2 u \\ x - c &= a \sin u \\ dx &= a \cos u \, du \\ a^2 - (x-c)^2 &= a^2 \cos^2 u \end{aligned}$$

Example 4.

$$\begin{aligned} &\int \sqrt{2 - x^2 + 4x} \, dx = \int \sqrt{2 + 4 - (x^2 + 4x + 4)} \, dx \\ &= \int \sqrt{6 - (x-2)^2} \, dx = 6 \int \cos^2 u \, du = \frac{6}{2} (u + \sin u \cos u) + C \\ &= 3 \left(\sin^{-1} \frac{x-2}{\sqrt{6}} + \frac{x-2}{6} \sqrt{6 - (x-2)^2} \right) + C \end{aligned}$$

$$x - 2 = \sqrt{6} \sin u, \quad dx = \sqrt{6} \cos u \, du, \quad 6 - (x-2)^2 = 6 \cos^2 u$$

Sine Substitution: $\sqrt{a^2 - x^2}$



Sine Substitution: $\sqrt{a^2 - x^2}$

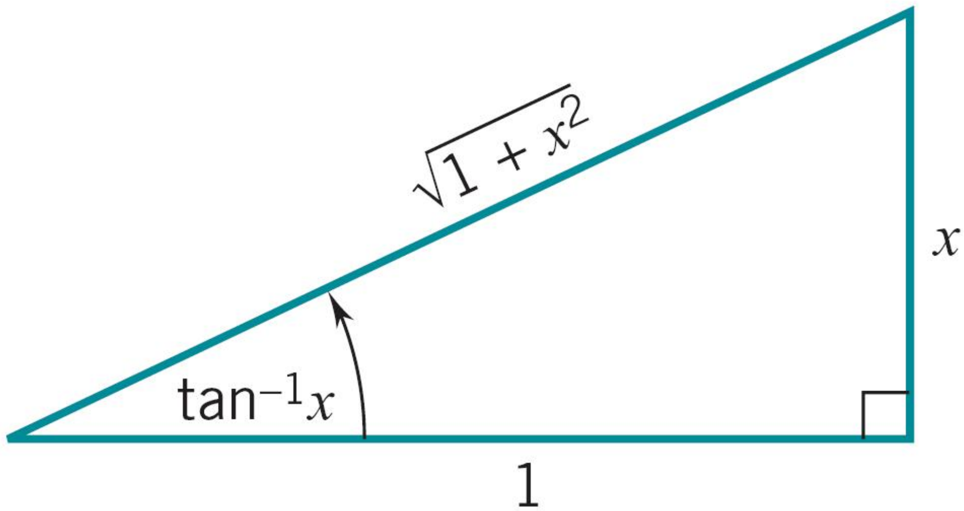
$$\begin{aligned} 1 - \sin^2 u &= \cos^2 u \\ x &= a \sin u \\ dx &= a \cos u \, du \\ a^2 - x^2 &= a^2 \cos^2 u \end{aligned}$$

Example 5.

$$\begin{aligned} \int \frac{\sqrt{4 - x^2}}{x} \, dx &= \int \frac{\sqrt{4 \cos^2 u}}{2 \sin u} 2 \cos u \, du = 2 \int \frac{\cos^2 u}{\sin u} \, du \\ &= 2 \int \frac{1 - \sin^2 u}{\sin u} \, du = 2 \int \csc u - \sin u \, du \\ &= 2 \ln |\csc u - \cot u| + 2 \cos u + C \\ &= 2 \ln |\frac{2}{x} - \frac{\sqrt{4 - x^2}}{x}| + \sqrt{4 - x^2} + C = \cdots \end{aligned}$$

1.2 Tangent Substitution

Tangent Substitution: $1 + x^2$



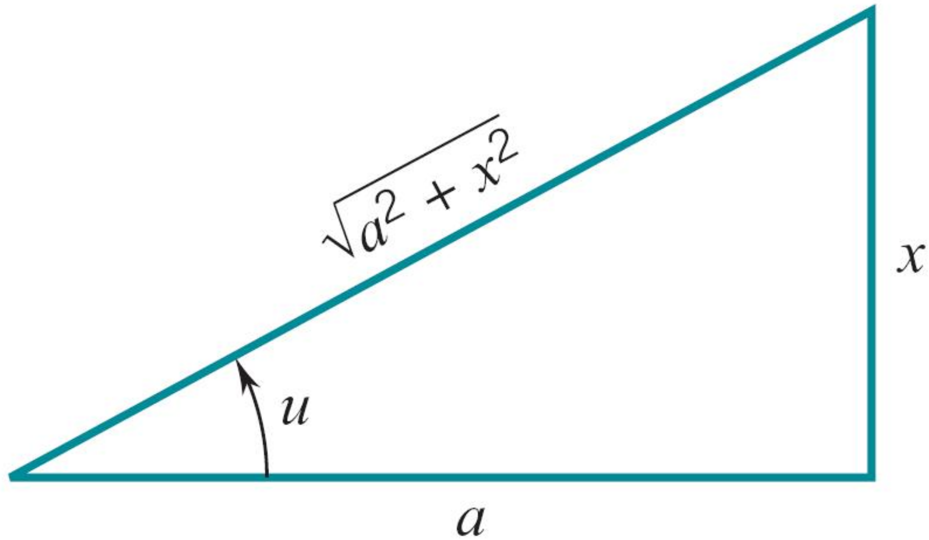
Tangent Substitution: $1+x^2$

$$\begin{aligned} 1+\tan^2 u &= \sec^2 u \\ x &= \tan u \\ dx &= \sec^2 u \, du \\ 1+x^2 &= \sec^2 u \end{aligned}$$

Example 6.

$$\begin{aligned} \int \frac{1}{(1+x^2)^2}\, dx &= \int \frac{1}{\sec^4 u} \sec^2 u \, du = \int \cos^2 u \, du \\ &= \int \left(\frac{1}{2} + \frac{1}{2} \cos 2u \right) \, du = \frac{1}{2} u + \frac{1}{4} \sin 2u + C \\ &= \frac{1}{2} \left(u + \sin u \cos u \right) + C \\ &= \frac{1}{2} \left(\tan^{-1} x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} \right) + C = \cdots \end{aligned}$$

Tangent Substitution: $\sqrt{a^2+x^2}$



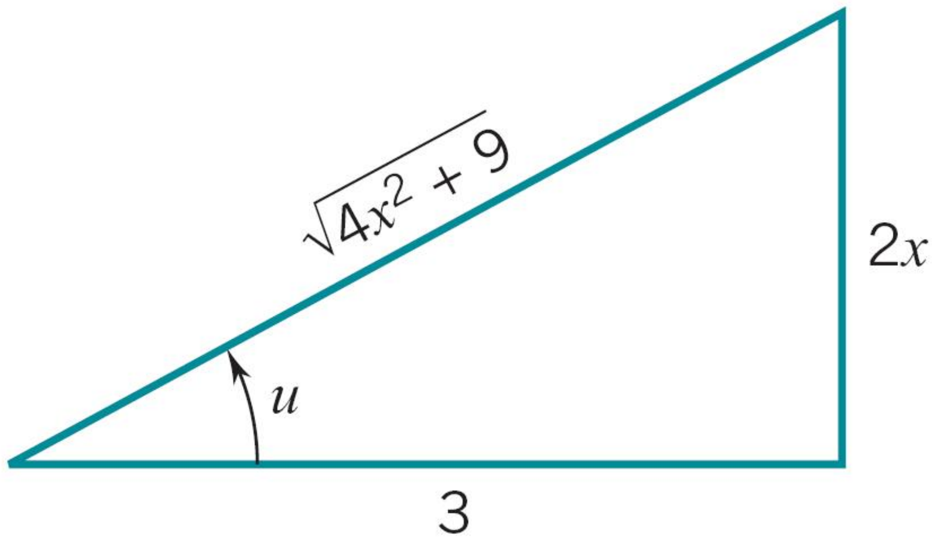
Tangent Substitution: $\sqrt{a^2+x^2}$

$$\begin{aligned} 1+\tan^2 u &= \sec^2 u \\ x &= a\tan u \\ dx &= a\sec^2 u\, du \\ a^2+x^2 &= a^2\sec^2 u \end{aligned}$$

Example 7.

$$\begin{aligned} \int \sqrt{a^2+x^2}\,dx &= \int \sqrt{a^2\sec^2 u} a\sec^2 u\,du = a^2\int \sec^3 u\,du \\ &= \frac{a^2}{2}\left(\sec u\tan u + \ln|\sec u + \tan u|\right) + C \\ &= \frac{a^2}{2}\left(\frac{\sqrt{a^2+x^2}}{a}\frac{x}{a} + \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|\right) + C = \cdots \end{aligned}$$

Example



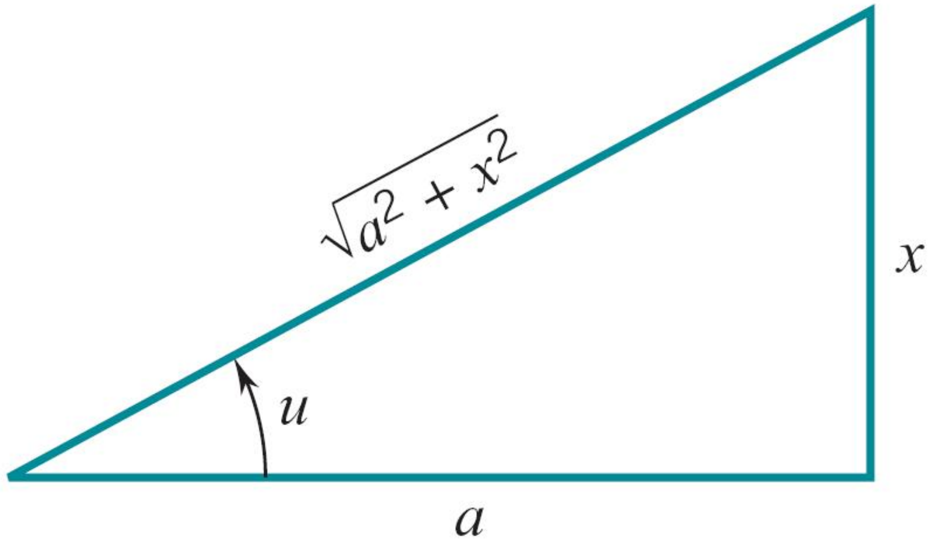
Tangent Substitution: $\sqrt{a^2+x^2}$

$$\begin{aligned} 1+\tan^2 u &= \sec^2 u \\ x &= a\tan u \\ dx &= a\sec^2 u\,du \\ a^2+x^2 &= a^2\sec^2 u \end{aligned}$$

Example 8.

$$\begin{aligned} \int \frac{1}{x\sqrt{9+4x^2}}\,dx &= \frac{1}{2}\int \frac{1}{\frac{3}{2}\tan u\sqrt{\frac{9}{4}\sec^2 u}}\frac{3}{2}\sec^2 u\,du \\ &= \frac{1}{3}\int \frac{\sec u}{\tan u}\,du = \frac{1}{3}\int \csc u\,du \\ &= \frac{1}{3}\ln|\csc u - \cot u| + C \\ &= \frac{1}{3}\ln|\frac{\sqrt{9+4x^2}-3}{2x}| + C \end{aligned}$$

Example



Tangent Substitution: $\sqrt{a^2 + x^2}$

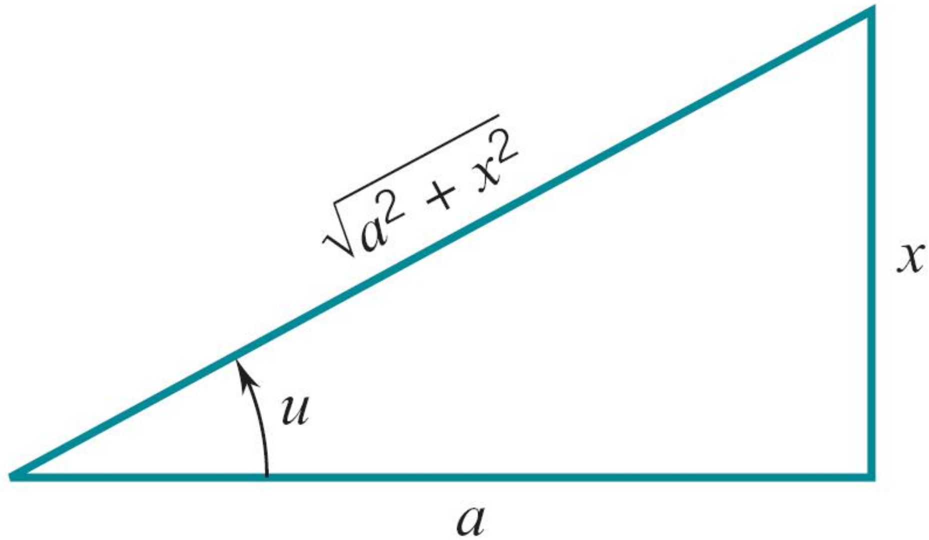
$$\begin{aligned} 1 + \tan^2 u &= \sec^2 u \\ x &= a \tan u \\ dx &= a \sec^2 u \, du \\ a^2 + x^2 &= a^2 \sec^2 u \end{aligned}$$

Example 9.

$$\begin{aligned} \int \frac{x^2}{\sqrt{4 + x^2}} \, dx &= \int \frac{4 \tan^2 u}{\sqrt{4 + 4 \tan^2 u}} 2 \sec^2 u \, du = \int \frac{8 \tan^2 u \sec^2 u}{\sqrt{4 \sec^2 u}} \, du \\ &= \int \frac{8 \tan^2 u \sec^2 u}{2 \sec u} \, du = 4 \int \tan^2 u \sec u \, du \end{aligned}$$

finish as in Example 12, Section 8.3
substitute in terms of x

Example: Completing the Square



Tangent Substitution: $\sqrt{a^2 + (x-c)^2}$

$$\begin{aligned} 1 + \tan^2 u &= \sec^2 u \\ x - c &= a \tan u \\ dx &= a \sec^2 u \, du \\ a^2 + (x-c)^2 &= a^2 \sec^2 u \end{aligned}$$

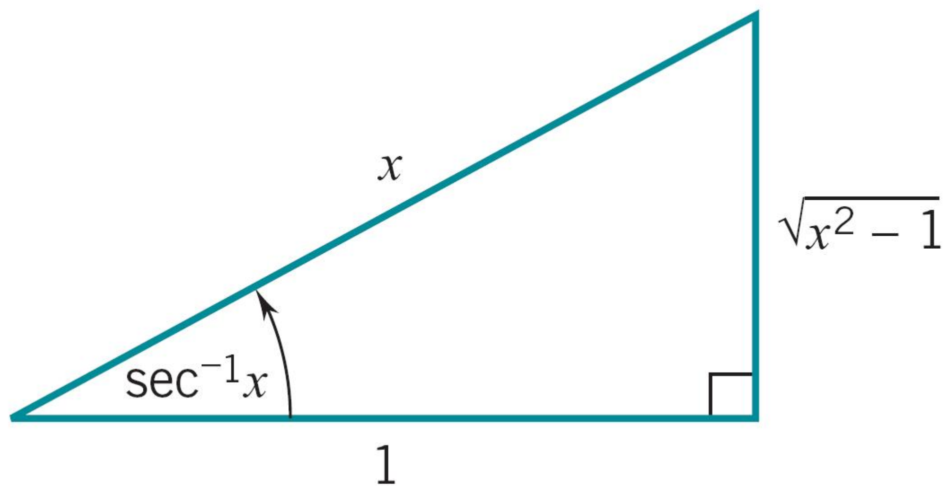
Example 10.

$$\int x\sqrt{16 + x^2 - 6x}\, dx = \int x\sqrt{16 - 9 + (x^2 - 6x + 9)}\, dx$$
$$= \int x\sqrt{7 + (x-3)^2}\, dx$$
$$= \int \left(\sqrt{7}\tan u + 3\right) \sqrt{7\sec^2 u}\sqrt{7}\sec^2 u \, du$$

Finish it!

1.3 Secant Substitution

Secant Substitution: $x^2 - 1$



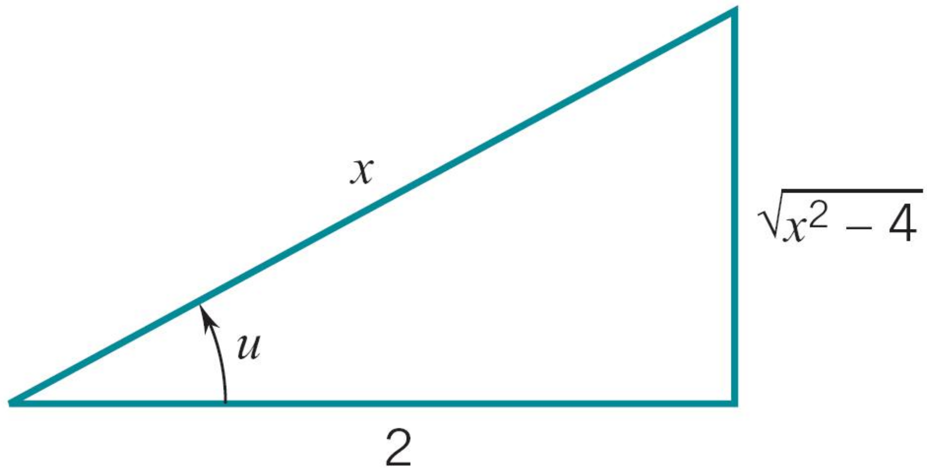
Secant Substitution: $x^2 - 1$

$$\begin{aligned} \sec^2 u - 1 &= \tan^2 u \\ x &= \sec u \\ dx &= \sec u \tan u \, du \\ x^2 - 1 &= \tan^2 u \end{aligned}$$

Example 11.

$$\begin{aligned} \int \frac{1}{x^2\sqrt{x^2-1}}\,dx &= \int \frac{\sec u \tan u}{\sec^2 u \sqrt{\tan^2 u}}\,du = \int \frac{1}{\sec u}\,du \\ &= \int \cos u \, du = \sin u + C \\ &= \frac{\sqrt{x^2-1}}{x} + C \end{aligned}$$

Secant Substitution: $\sqrt{x^2 - a^2}$



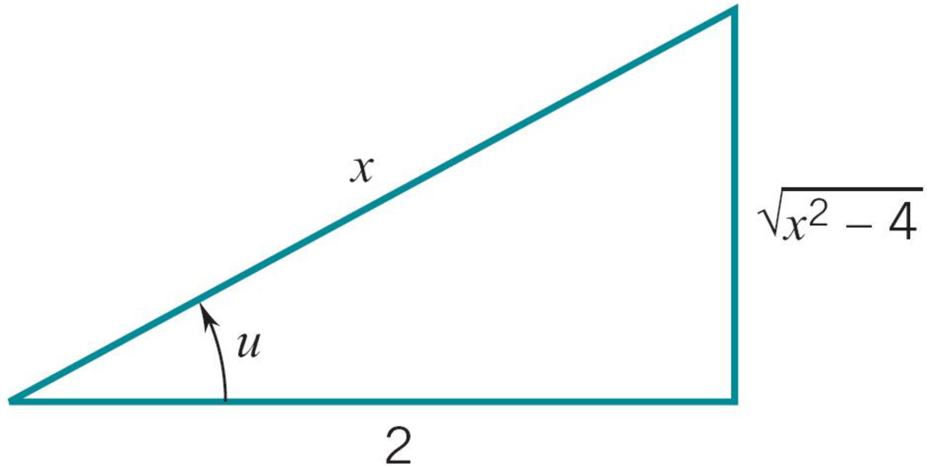
Secant Substitution: $\sqrt{x^2 - a^2}$

$$\begin{aligned} \sec^2 u - 1 &= \tan^2 u \\ x &= a \sec u \\ dx &= a \sec u \tan u \, du \\ x^2 - a^2 &= a^2 \tan^2 u \end{aligned}$$

Example 12.

$$\begin{aligned} \int \frac{1}{x(x^2-4)^{3/2}} \, dx = \int \frac{2 \sec u \tan u}{2 \sec u (4 \tan^2 u)^{3/2}} \, du &= \frac{1}{8} \int \frac{1}{\tan^2 u} \, du \\ &= \frac{1}{8} \int \cot^2 u \, du = \frac{1}{8} \int \left(\csc^2 u - 1 \right) \, du \\ &= \frac{1}{8} \left(-\cot u - u \right) + C \\ &= -\frac{1}{8} \left(\frac{2}{\sqrt{x^2-4}} + \sec^{-1}(x/2) \right) + C \end{aligned}$$

Example: Completing the Square



Secant Substitution: $\sqrt{x^2 - a^2}$

$$\begin{aligned} \sec^2 u - 1 &= \tan^2 u \\ x - c &= a \sec u \\ dx &= a \sec u \tan u \, du \\ (x-c)^2 - a^2 &= a^2 \tan^2 u \end{aligned}$$

Example 13.

$$\int \frac{x}{\sqrt{x^2+2x-3}}\, dx = \int \frac{x}{\sqrt{(x^2+2x+1)-316-1}}\, dx$$

$$= \int \frac{x}{\sqrt{(x+1)^2-4}}, dx$$

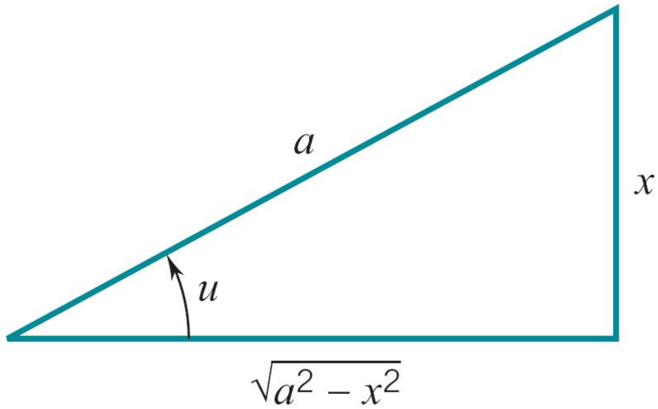
$$= \int \frac{2\sec u - 1}{\sqrt{4\tan^2 u}} 2 \sec u \tan u \, du$$

Finish it!

Summary

$a^2 - x^2$

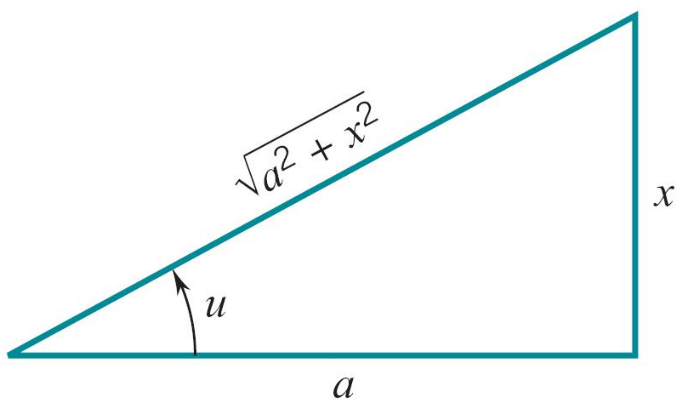
$$\begin{aligned} 1 - \sin^2 u &= \cos^2 u \\ x &= a \sin u \\ dx &= a \cos u \, du \\ a^2 - x^2 &= a^2 \cos^2 u \end{aligned}$$



$$\sin u = \frac{x}{a}$$
$$\cos u = \frac{\sqrt{a^2 - x^2}}{a}$$
$$\tan u = \frac{x}{\sqrt{a^2 - x^2}}$$

$a^2 + x^2$

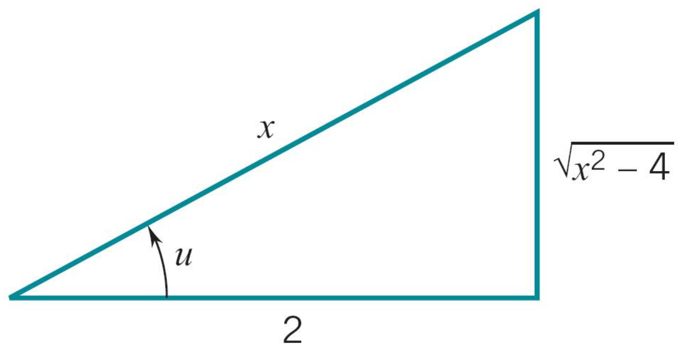
$$1 + \tan^2 u = \sec^2 u$$
$$x = a \tan u$$
$$dx = a \sec^2 u \, du$$
$$a^2 + x^2 = a^2 \sec^2 u$$



$$\sin u = \frac{x}{\sqrt{a^2 + x^2}}$$
$$\cos u = \frac{a}{\sqrt{a^2 + x^2}}$$
$$\tan u = \frac{x}{a}$$

$x^2 - a^2$

$$\sec^2 u - 1 = \tan^2 u$$
$$x = a \sec u$$
$$dx = a \sec u \tan u \, du$$
$$x^2 - a^2 = a^2 \tan^2 u$$



$$\sin u = \frac{\sqrt{x^2 - a^2}}{x}$$
$$\cos u = \frac{a}{x}$$
$$\tan u = \frac{\sqrt{x^2 - a^2}}{a}$$

Outline

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