Lecture 10

Section 8.5 Rational Functions; Partial Fractions

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
http://math.uh.edu/∼jiwenhe/Math1432

$$\int \frac{A}{(x-\alpha)^k}\,dx, \quad \int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx$$

Rational Function

- **Rational function**: $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
  - Yes: $\dfrac{2x}{x^2 - x - 2}, \quad \dfrac{3x^4 - 20x^2 + 17}{x^3 + 2x^2 - 7}$
  - No: $\dfrac{1}{\sqrt{x}}, \quad \dfrac{x^2 + 1}{\ln x}$
- If degree$(P) \geq$ degree(Q), then, by division,
$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}$$
where $p(x)$ is a polynomial and $\frac{r(x)}{Q(x)}$ is a proper rational function (i.e., degree$(r) <$ degree(Q)).
- $\dfrac{x^2}{x^2 - 2x - 3} = 1 + \dfrac{2x + 3}{x^2 - 2x - 3}$
- $\dfrac{x^3}{x^2 - 2x + 1} = x + 2 + \dfrac{3x - 2}{x^2 - 2x + 1}$

Rational Function

- Rational function: $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
  - Yes: $\dfrac{2x}{x^2 - x - 2}, \quad \dfrac{3x^4 - 20x^2 + 17}{x^3 + 2x^2 - 7}$
  - No: $\dfrac{1}{\sqrt{x}}, \quad \dfrac{x^2 + 1}{\ln x}$
- If degree$(P) \geq$ degree(Q), then, by division,
$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}$$
where $p(x)$ is a polynomial and $\frac{r(x)}{Q(x)}$ is a proper rational function (i.e., degree$(r) <$ degree(Q)).
- $\dfrac{x^2}{x^2 - 2x - 3} = 1 + \dfrac{2x + 3}{x^2 - 2x - 3}$
- $\dfrac{x^3}{x^2 - 2x + 1} = x + 2 + \dfrac{3x - 2}{x^2 - 2x + 1}$

Rational Function

- Rational function: $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
  - Yes: $\dfrac{2x}{x^2 - x - 2}$, $\dfrac{3x^4 - 20x^2 + 17}{x^3 + 2x^2 - 7}$
  - No: $\dfrac{1}{\sqrt{x}}$, $\dfrac{x^2 + 1}{\ln x}$
- If degree(P) $\geq$ degree(Q), then, by division,

$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}$$

  where $p(x)$ is a polynomial and $\frac{r(x)}{Q(x)}$ is a proper rational function (i.e., degree(r) $<$ degree(Q)).
- $\dfrac{x^2}{x^2 - 2x - 3} = 1 + \dfrac{2x + 3}{x^2 - 2x - 3}$
- $\dfrac{x^3}{x^2 - 2x + 1} = x + 2 + \dfrac{3x - 2}{x^2 - 2x + 1}$

Rational Function

- Rational function: $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
  - Yes: $\dfrac{2x}{x^2 - x - 2}$, $\dfrac{3x^4 - 20x^2 + 17}{x^3 + 2x^2 - 7}$
  - No: $\dfrac{1}{\sqrt{x}}$, $\dfrac{x^2+1}{\ln x}$
- If degree$(P) \geq$ degree(Q), then, by division,

$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}$$

  where $p(x)$ is a polynomial and $\frac{r(x)}{Q(x)}$ is a proper rational function (i.e., degree$(r) <$ degree(Q)).
- $\dfrac{x^2}{x^2 - 2x - 3} = 1 + \dfrac{2x+3}{x^2 - 2x - 3}$
- $\dfrac{x^3}{x^2 - 2x + 1} = x + 2 + \dfrac{3x-2}{x^2 - 2x + 1}$

Rational Function

- Rational function: $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
  - Yes: $\dfrac{2x}{x^2 - x - 2}$, $\dfrac{3x^4 - 20x^2 + 17}{x^3 + 2x^2 - 7}$
  - No: $\dfrac{1}{\sqrt{x}}$, $\dfrac{x^2 + 1}{\ln x}$
- If degree$(P) \geq$ degree(Q), then, by division,

$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}$$

where $p(x)$ is a polynomial and $\frac{r(x)}{Q(x)}$ is a proper rational function (i.e., degree$(r) <$ degree(Q)).

- $\dfrac{x^2}{x^2 - 2x - 3} = 1 + \dfrac{2x + 3}{x^2 - 2x - 3}$
- $\dfrac{x^3}{x^2 - 2x + 1} = x + 2 + \dfrac{3x - 2}{x^2 - 2x + 1}$

Rational Function

- Rational function: $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
  - Yes: $\dfrac{2x}{x^2 - x - 2}$, $\dfrac{3x^4 - 20x^2 + 17}{x^3 + 2x^2 - 7}$
  - No: $\dfrac{1}{\sqrt{x}}$, $\dfrac{x^2+1}{\ln x}$
- If degree$(P) \geq$ degree(Q), then, by division,
$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}$$
where $p(x)$ is a polynomial and $\frac{r(x)}{Q(x)}$ is a proper rational function (i.e., degree$(r) <$ degree(Q)).
- $\dfrac{x^2}{x^2 - 2x - 3} = 1 + \dfrac{2x + 3}{x^2 - 2x - 3}$
- $\dfrac{x^3}{x^2 - 2x + 1} = x + 2 + \dfrac{3x - 2}{x^2 - 2x + 1}$

Rational Function

- Rational function: $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
  - Yes: $\frac{2x}{x^2 - x - 2}$, $\frac{3x^4 - 20x^2 + 17}{x^3 + 2x^2 - 7}$
  - No: $\frac{1}{\sqrt{x}}$, $\frac{x^2 + 1}{\ln x}$
- If degree$(P) \geq$ degree(Q), then, by division,
$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}$$
where $p(x)$ is a polynomial and $\frac{r(x)}{Q(x)}$ is a proper rational function (i.e., degree$(r) <$ degree(Q)).
- $\frac{x^2}{x^2 - 2x - 3} = 1 + \frac{2x + 3}{x^2 - 2x - 3}$
- $\frac{x^3}{x^2 - 2x + 1} = x + 2 + \frac{3x - 2}{x^2 - 2x + 1}$

Partial Fraction Decomposition

Partial Fraction Decomposition

- Let the denominator $Q(x)$ factor as
$$Q(x) = a\prod(x-\alpha_i)^{n_i}\prod(x^2+\beta_j x+\gamma_j)^{n_j}$$
where the quadratic factors $x^2+\beta_j x+\gamma_j$ are irreducible (i.e., $\beta_j^2-4\gamma_j<0$, they have complex zeros).
- Proper rational function $R(x)=\frac{P(x)}{Q(x)}$ can be written as a sum of partial fractions of the form:
$$\frac{A}{(x-\alpha)^k},\quad \frac{Bx+C}{(x^2+\beta x+\gamma)^k}$$
  - Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots+\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
  - Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots+\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Partial Fraction Decomposition

Partial Fraction Decomposition

- Let the denominator $Q(x)$ factor as
$$Q(x) = a\prod(x-\alpha_i)^{n_i}\prod(x^2+\beta_j x+\gamma_j)^{n_j}$$
where the quadratic factors $x^2+\beta_j x+\gamma_j$ are irreducible (i.e., $\beta_j^2-4\gamma_j<0$, they have complex zeros).
- Proper rational function $R(x)=\frac{P(x)}{Q(x)}$ can be written as a sum of partial fractions of the form:
$$\frac{A}{(x-\alpha)^k},\quad \frac{Bx+C}{(x^2+\beta x+\gamma)^k}$$
  - Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
  - Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Partial Fraction Decomposition

Partial Fraction Decomposition

- Let the denominator $Q(x)$ factor as
$$Q(x) = a\prod(x-\alpha_i)^{n_i}\prod(x^2+\beta_j x+\gamma_j)^{n_j}$$
where the quadratic factors $x^2+\beta_j x+\gamma_j$ are irreducible (i.e., $\beta_j^2-4\gamma_j<0$, they have complex zeros).
- Proper rational function $R(x)=\frac{P(x)}{Q(x)}$ can be written as a sum of partial fractions of the form:
$$\frac{A}{(x-\alpha)^k},\quad \frac{Bx+C}{(x^2+\beta x+\gamma)^k}$$
  - Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
  - Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Partial Fraction Decomposition

Partial Fraction Decomposition

- Let the denominator $Q(x)$ factor as
$$Q(x) = a\prod(x-\alpha_i)^{n_i}\prod(x^2+\beta_j x+\gamma_j)^{n_j}$$
where the quadratic factors $x^2+\beta_j x+\gamma_j$ are irreducible (i.e., $\beta_j^2 - 4\gamma_j < 0$, they have complex zeros).
- Proper rational function $R(x) = \frac{P(x)}{Q(x)}$ can be written as a sum of partial fractions of the form:
$$\frac{A}{(x-\alpha)^k}, \quad \frac{Bx+C}{(x^2+\beta x+\gamma)^k}$$
  - Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n} + \cdots \frac{A_2}{(x-\alpha)^2} + \frac{A_1}{x-\alpha}$$
  - Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n} + \cdots \frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2} + \frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

:

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:

$$\frac{A_n}{(x-\alpha)^n} + \cdots \frac{A_2}{(x-\alpha)^2} + \frac{A_1}{x-\alpha}$$

- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:

$$\frac{B_n x + C_n}{(x^2+\beta x+\gamma)^n} + \cdots \frac{B_2 x + C_2}{(x^2+\beta x+\gamma)^2} + \frac{B_1 x + C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{1}{x^3(x^2+1)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{Dx+E}{x^2+1} = \frac{1}{x^3} - \frac{1}{x} + \frac{x}{x^2+1}$$

$$1 = (C+D)x^4 + (B+E)x^3 + (A+C)x^2 + Bx + A$$

$$A=1, \quad B=0, \quad A+C=0, \quad B+E=0, \quad C+D=0$$

$$A=1, \quad B=0, \quad C=-1, \quad E=0, \quad D=1$$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:

$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$

- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:

$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{1}{x^3(x^2+1)}=\frac{A}{x^3}+\frac{B}{x^2}+\frac{C}{x}+\frac{Dx+E}{x^2+1}=\frac{1}{x^3}-\frac{1}{x}+\frac{x}{x^2+1}$$

$$1=(C+D)x^4+(B+E)x^3+(A+C)x^2+Bx+A$$

$$A=1,\quad B=0,\quad A+C=0,\quad B+E=0,\quad C+D=0$$

$$A=1,\quad B=0,\quad C=-1,\quad E=0,\quad D=1$$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{1}{x^3(x^2+1)}=\frac{A}{x^3}+\frac{B}{x^2}+\frac{C}{x}+\frac{Dx+E}{x^2+1}=\frac{1}{x^3}-\frac{1}{x}+\frac{x}{x^2+1}$$

$$1=(C+D)x^4+(B+E)x^3+(A+C)x^2+Bx+A$$

$$A=1,\quad B=0,\quad A+C=0,\quad B+E=0,\quad C+D=0$$

$$A=1,\quad B=0,\quad C=-1,\quad E=0,\quad D=1$$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n} + \cdots \frac{A_2}{(x-\alpha)^2} + \frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x + C_n}{(x^2+\beta x+\gamma)^n} + \cdots \frac{B_2 x + C_2}{(x^2+\beta x+\gamma)^2} + \frac{B_1 x + C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{1}{x^3(x^2+1)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{Dx+E}{x^2+1} = \frac{1}{x^3} - \frac{1}{x} + \frac{x}{x^2+1}$$

$$1 = (C+D)x^4 + (B+E)x^3 + (A+C)x^2 + Bx + A$$
$$A=1, \quad B=0, \quad A+C=0, \quad B+E=0, \quad C+D=0$$
$$A=1, \quad B=0, \quad C=-1, \quad E=0, \quad D=1$$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{1}{x^3(x^2+1)}=\frac{A}{x^3}+\frac{B}{x^2}+\frac{C}{x}+\frac{Dx+E}{x^2+1}=\frac{1}{x^3}-\frac{1}{x}+\frac{x}{x^2+1}$$

$$1=(C+D)x^4+(B+E)x^3+(A+C)x^2+Bx+A$$
$$A=1,\quad B=0,\quad A+C=0,\quad B+E=0,\quad C+D=0$$
$$A=1,\quad B=0,\quad C=-1,\quad E=0,\quad D=1$$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{1}{x^3(x^2+1)}=\frac{A}{x^3}+\frac{B}{x^2}+\frac{C}{x}+\frac{Dx+E}{x^2+1}=\frac{1}{x^3}-\frac{1}{x}+\frac{x}{x^2+1}$$

$$1=(C+D)x^4+(B+E)x^3+(A+C)x^2+Bx+A$$
$$A=1,\quad B=0,\quad A+C=0,\quad B+E=0,\quad C+D=0$$
$$A=1,\quad B=0,\quad C=-1,\quad E=0,\quad D=1$$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:

$$\frac{A_n}{(x-\alpha)^n} + \cdots \frac{A_2}{(x-\alpha)^2} + \frac{A_1}{x-\alpha}$$

- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:

$$\frac{B_n x + C_n}{(x^2+\beta x+\gamma)^n} + \cdots \frac{B_2 x + C_2}{(x^2+\beta x+\gamma)^2} + \frac{B_1 x + C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{x^2}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+1)^2} + \frac{Dx+E}{x^2+1}$$

$$\Rightarrow \quad x^2 = (A+D)x^4 + (-D+E)x^3 + (2A+B+D-E)x^2 + (-B+C-D+E)x + (A-C-E)$$

$$\Rightarrow \quad A+D=0, \quad -D+E=0, \quad 2A+B+D-E=1, \quad -B+C-D+E=0, \quad A-C-E=0 \quad \Rightarrow \text{Finish it}$$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{x^2}{(x-1)(x^2+1)^2}=\frac{A}{x-1}+\frac{Bx+C}{(x^2+1)^2}+\frac{Dx+E}{x^2+1}$$

$$\Rightarrow \quad x^2=(A+D)x^4+(-D+E)x^3+(2A+B+D-E)x^2 + (-B+C-D+E)x+(A-C-E)$$

$$\Rightarrow \quad A+D=0, \quad -D+E=0, \quad 2A+B+D-E=1, \quad -B+C-D+E=0, \quad A-C-E=0 \quad \Rightarrow \text{Finish it}$$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n} + \cdots \frac{A_2}{(x-\alpha)^2} + \frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x + C_n}{(x^2+\beta x+\gamma)^n} + \cdots \frac{B_2 x + C_2}{(x^2+\beta x+\gamma)^2} + \frac{B_1 x + C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{x^2}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+1)^2} + \frac{Dx+E}{x^2+1}$$

$$\Rightarrow \quad x^2 = (A+D)x^4 + (-D+E)x^3 + (2A+B+D-E)x^2 + (-B+C-D+E)x + (A-C-E)$$

$$\Rightarrow \quad A+D=0, \quad -D+E=0, \quad 2A+B+D-E=1, \quad -B+C-D+E=0, \quad A-C-E=0 \quad \Rightarrow \text{Finish it}$$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{x^2}{(x-1)(x^2+1)^2}=\frac{A}{x-1}+\frac{Bx+C}{(x^2+1)^2}+\frac{Dx+E}{x^2+1}$$

$$\Rightarrow \quad x^2=(A+D)x^4+(-D+E)x^3+(2A+B+D-E)x^2$$
$$+(-B+C-D+E)x+(A-C-E)$$

$$\Rightarrow \quad A+D=0, \quad -D+E=0, \quad 2A+B+D-E=1,$$
$$-B+C-D+E=0, \quad A-C-E=0 \quad \Rightarrow \text{Finish it}$$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{x^2}{(x-1)(x^2+1)^2}=\frac{A}{x-1}+\frac{Bx+C}{(x^2+1)^2}+\frac{Dx+E}{x^2+1}$$

$$\Rightarrow \quad x^2=(A+D)x^4+(-D+E)x^3+(2A+B+D-E)x^2 + (-B+C-D+E)x+(A-C-E)$$

$$\Rightarrow \quad A+D=0, \quad -D+E=0, \quad 2A+B+D-E=1, \quad -B+C-D+E=0, \quad A-C-E=0 \quad \Rightarrow \text{Finish it}$$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{x^3}{(x^2+2x+2)^2}=\frac{Ax+B}{(x^2+2x+2)^2}+\frac{Cx+D}{x^2+2x+2}$$

$$\Rightarrow \quad x^3=Cx^3+(2C+D)x^2+(A+2C+2D)x+(B+2D)$$

$$\Rightarrow \quad C=1, \quad 2C+D=0, \quad A+2C+2D=0, \quad B+2D=0$$

$\Rightarrow$ Finish it

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:

$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$

- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:

$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{x^3}{(x^2+2x+2)^2}=\frac{Ax+B}{(x^2+2x+2)^2}+\frac{Cx+D}{x^2+2x+2}$$

$\Rightarrow \quad x^3 = Cx^3+(2C+D)x^2+(A+2C+2D)x+(B+2D)$

$\Rightarrow \quad C=1, \quad 2C+D=0, \quad A+2C+2D=0, \quad B+2D=0$

$\Rightarrow$ Finish it

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{x^3}{(x^2+2x+2)^2}=\frac{Ax+B}{(x^2+2x+2)^2}+\frac{Cx+D}{x^2+2x+2}$$

$$\Rightarrow \quad x^3=Cx^3+(2C+D)x^2+(A+2C+2D)x+(B+2D)$$

$$\Rightarrow \quad C=1,\quad 2C+D=0,\quad A+2C+2D=0,\quad B+2D=0$$

$\Rightarrow$ Finish it

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{x^3}{(x^2+2x+2)^2}=\frac{Ax+B}{(x^2+2x+2)^2}+\frac{Cx+D}{x^2+2x+2}$$

$$\Rightarrow \quad x^3=Cx^3+(2C+D)x^2+(A+2C+2D)x+(B+2D)$$

$$\Rightarrow \quad C=1,\quad 2C+D=0,\quad A+2C+2D=0,\quad B+2D=0$$

$\Rightarrow$ Finish it

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes:
$$\frac{A_n}{(x-\alpha)^n}+\cdots\frac{A_2}{(x-\alpha)^2}+\frac{A_1}{x-\alpha}$$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes:
$$\frac{B_n x+C_n}{(x^2+\beta x+\gamma)^n}+\cdots\frac{B_2 x+C_2}{(x^2+\beta x+\gamma)^2}+\frac{B_1 x+C_1}{x^2+\beta x+\gamma}$$

Examples

$$\frac{x^3}{(x^2+2x+2)^2}=\frac{Ax+B}{(x^2+2x+2)^2}+\frac{Cx+D}{x^2+2x+2}$$

$$\Rightarrow \quad x^3=Cx^3+(2C+D)x^2+(A+2C+2D)x+(B+2D)$$

$$\Rightarrow \quad C=1, \quad 2C+D=0, \quad A+2C+2D=0, \quad B+2D=0$$

$\Rightarrow$ Finish it

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\int \frac{6}{x^3-5x^2+6x}\,dx = \int \frac{6}{x(x-2)(x-3)}\,dx$$

$$= \int \left(\frac{1}{x} - \frac{3}{x-2} + \frac{2}{x-3}\right)dx$$

$$= \ln|x| - 3\ln|x-2| + 2\ln|x-3| + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\int \frac{6}{x^3-5x^2+6x}\,dx = \int \frac{6}{x(x-2)(x-3)}\,dx$$

$$= \int \left(\frac{1}{x} - \frac{3}{x-2} + \frac{2}{x-3}\right) dx$$

$$= \ln|x| - 3\ln|x-2| + 2\ln|x-3| + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\int \frac{6}{x^3-5x^2+6x}\,dx = \int \frac{6}{x(x-2)(x-3)}\,dx$$

$$= \int \left(\frac{1}{x} - \frac{3}{x-2} + \frac{2}{x-3}\right)dx$$

$$= \ln|x| - 3\ln|x-2| + 2\ln|x-3| + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\int \frac{6}{x^3-5x^2+6x}\,dx = \int \frac{6}{x(x-2)(x-3)}\,dx$$

$$= \int \left(\frac{1}{x} - \frac{3}{x-2} + \frac{2}{x-3}\right) dx$$

$$= \ln|x| - 3\ln|x-2| + 2\ln|x-3| + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\int \frac{1}{x^3-2x^2}\,dx = \int \frac{1}{x^2(x-2)}\,dx$$

$$= \int \left(\frac{-1/2}{x^2} + \frac{-1/4}{x} + \frac{1/4}{x-2}\right) dx$$

$$= \frac{1}{4}\left(\frac{2}{x} - \ln|x| + \ln|x-2|\right) + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\int \frac{1}{x^3-2x^2}\,dx = \int \frac{1}{x^2(x-2)}\,dx$$

$$= \int \left(\frac{-1/2}{x^2} + \frac{-1/4}{x} + \frac{1/4}{x-2}\right)dx$$

$$= \frac{1}{4}\left(\frac{2}{x} - \ln|x| + \ln|x-2|\right) + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\int \frac{1}{x^3-2x^2}\,dx = \int \frac{1}{x^2(x-2)}\,dx$$

$$= \int \left(\frac{-1/2}{x^2} + \frac{-1/4}{x} + \frac{1/4}{x-2}\right)\,dx$$

$$= \frac{1}{4}\left(\frac{2}{x} - \ln|x| + \ln|x-2|\right) + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\begin{aligned}\int \frac{1}{x^3-2x^2}\,dx &= \int \frac{1}{x^2(x-2)}\,dx \\ &= \int \left(\frac{-1/2}{x^2} + \frac{-1/4}{x} + \frac{1/4}{x-2}\right) dx \\ &= \frac{1}{4}\left(\frac{2}{x} - \ln|x| + \ln|x-2|\right) + C\end{aligned}$$

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\int \frac{x^2}{(x^2-9)^2}\,dx = \int \frac{x^2}{(x-3)^2(x+3)^2}\,dx$$

$$= \int \left(\frac{1/4}{(x-3)^2} + \frac{1/12}{x-3} + \frac{1/4}{(x+3)^2} + \frac{-1/12}{x+3} \right) dx$$

$$= \frac{1}{12}\left(-\frac{3}{x-3} + \ln|x-3| - \frac{3}{x+3} - \ln|x+3| \right) + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\int \frac{x^2}{(x^2-9)^2}\,dx = \int \frac{x^2}{(x-3)^2(x+3)^2}\,dx$$

$$= \int \left(\frac{1/4}{(x-3)^2} + \frac{1/12}{x-3} + \frac{1/4}{(x+3)^2} + \frac{-1/12}{x+3}\right) dx$$

$$= \frac{1}{12}\left(-\frac{3}{x-3} + \ln|x-3| - \frac{3}{x+3} - \ln|x+3|\right) + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\int \frac{x^2}{(x^2-9)^2}\,dx = \int \frac{x^2}{(x-3)^2(x+3)^2}\,dx$$

$$= \int \left(\frac{1/4}{(x-3)^2} + \frac{1/12}{x-3} + \frac{1/4}{(x+3)^2} + \frac{-1/12}{x+3} \right) dx$$

$$= \frac{1}{12}\left(-\frac{3}{x-3} + \ln|x-3| - \frac{3}{x+3} - \ln|x+3| \right) + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$

$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples

$$\int \frac{x^2}{(x^2-9)^2}\,dx = \int \frac{x^2}{(x-3)^2(x+3)^2}\,dx$$

$$= \int \left(\frac{1/4}{(x-3)^2} + \frac{1/12}{x-3} + \frac{1/4}{(x+3)^2} + \frac{-1/12}{x+3}\right) dx$$

$$= \frac{1}{12}\left(-\frac{3}{x-3} + \ln|x-3| - \frac{3}{x+3} - \ln|x+3|\right) + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|x^2+\beta x+\gamma\right| + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|x^2+\beta x+\gamma\right| + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|x^2+\beta x+\gamma\right| + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|x^2+\beta x+\gamma\right| + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|x^2+\beta x+\gamma\right| + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|x^2+\beta x+\gamma\right| + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|x^2+\beta x+\gamma\right| + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|x^2+\beta x+\gamma\right| + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{1}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{t^2+a^2}\,dt = \int \frac{1}{a^2\sec^2 u}\,a\sec^2 u\,du$$

$$= \frac{1}{a}\int du = \frac{1}{a}u + C = \frac{1}{a}\tan^{-1}\frac{t}{a} = \frac{1}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.

Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.

set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{1}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{t^2+a^2}\,dt = \int \frac{1}{a^2\sec^2 u}\,a\sec^2 u\,du$$

$$= \frac{1}{a}\int du = \frac{1}{a}u + C = \frac{1}{a}\tan^{-1}\frac{t}{a} = \frac{1}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.
Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.
set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{1}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{t^2+a^2}\,dt = \int \frac{1}{a^2\sec^2 u}\,a\sec^2 u\,du$$

$$= \frac{1}{a}\int du = \frac{1}{a}u + C = \frac{1}{a}\tan^{-1}\frac{t}{a} = \frac{1}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.

Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.

set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{1}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{t^2+a^2}\,dt = \int \frac{1}{a^2\sec^2 u}\,a\sec^2 u\,du$$

$$= \frac{1}{a}\int du = \frac{1}{a}u + C = \frac{1}{a}\tan^{-1}\frac{t}{a} = \frac{1}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.
Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.
set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{1}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{t^2+a^2}\,dt = \int \frac{1}{a^2\sec^2 u}\,a\sec^2 u\,du$$

$$= \frac{1}{a}\int du = \frac{1}{a}u + C = \frac{1}{a}\tan^{-1}\frac{t}{a} = \frac{1}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.

Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.

set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{1}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{t^2+a^2}\,dt = \int \frac{1}{a^2\sec^2 u}\,a\sec^2 u\,du$$

$$= \frac{1}{a}\int du = \frac{1}{a}u + C = \frac{1}{a}\tan^{-1}\frac{t}{a} = \frac{1}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.
Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.
set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{1}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{t^2+a^2}\,dt = \int \frac{1}{a^2\sec^2 u}\,a\sec^2 u\,du$$

$$= \frac{1}{a}\int du = \frac{1}{a}u + C = \frac{1}{a}\tan^{-1}\frac{t}{a} = \frac{1}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.

Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.

set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{1}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{t^2+a^2}\,dt = \int \frac{1}{a^2\sec^2 u}\,a\sec^2 u\,du$$

$$= \frac{1}{a}\int du = \frac{1}{a}u + C = \frac{1}{a}\tan^{-1}\frac{t}{a} = \frac{1}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.
Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.
set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{x^2}{(x+1)(x^2+4)}\,dx = \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x-4}{x^2+4}\right)dx$$

$$= \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x}{x^2+4}-\frac{4}{x^2+4}\right)dx$$

$$= \frac{1}{5}\left(\ln|x+1| + 2\ln\left(x^2+4\right) - 2\tan^{-1}\frac{x}{2}\right) + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{x^2}{(x+1)(x^2+4)}\,dx = \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x-4}{x^2+4}\right)dx$$

$$= \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x}{x^2+4}-\frac{4}{x^2+4}\right)dx$$

$$= \frac{1}{5}\left(\ln|x+1| + 2\ln\left(x^2+4\right) - 2\tan^{-1}\frac{x}{2}\right) + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{x^2}{(x+1)(x^2+4)}\,dx = \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x-4}{x^2+4}\right)dx$$

$$= \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x}{x^2+4}-\frac{4}{x^2+4}\right)dx$$

$$= \frac{1}{5}\left(\ln|x+1| + 2\ln\left(x^2+4\right) - 2\tan^{-1}\frac{x}{2}\right) + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{x^2}{(x+1)(x^2+4)}\,dx = \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x-4}{x^2+4}\right)dx$$

$$= \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x}{x^2+4}-\frac{4}{x^2+4}\right)dx$$

$$= \frac{1}{5}\left(\ln|x+1| + 2\ln\left(x^2+4\right) - 2\tan^{-1}\frac{x}{2}\right) + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\begin{aligned}
\int \frac{x^2}{(x+1)(x^2+4)}\,dx &= \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x-4}{x^2+4}\right)dx \\
&= \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x}{x^2+4}-\frac{4}{x^2+4}\right)dx \\
&= \frac{1}{5}\left(\ln|x+1| + 2\ln\left(x^2+4\right) - 2\tan^{-1}\frac{x}{2}\right) + C
\end{aligned}$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{x^2+5x+2}{(x+1)(x^2+1)}\,dx = \int\left(\frac{-1}{x+1}+\frac{2x+3}{x^2+1}\right)dx$$

$$= \int\left(-\frac{1}{x+1}+\frac{2x}{x^2+1}+\frac{3}{x^2+1}\right)dx$$

$$= -\ln|x+1| + \ln\left(x^2+1\right) + 3\tan^{-1}x + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{x^2+5x+2}{(x+1)(x^2+1)}\,dx = \int\left(\frac{-1}{x+1}+\frac{2x+3}{x^2+1}\right)dx$$

$$= \int\left(-\frac{1}{x+1}+\frac{2x}{x^2+1}+\frac{3}{x^2+1}\right)dx$$

$$= -\ln|x+1| + \ln\left(x^2+1\right) + 3\tan^{-1}x + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{x^2+5x+2}{(x+1)(x^2+1)}\,dx = \int\left(\frac{-1}{x+1}+\frac{2x+3}{x^2+1}\right)dx$$

$$= \int\left(-\frac{1}{x+1}+\frac{2x}{x^2+1}+\frac{3}{x^2+1}\right)dx$$

$$= -\ln|x+1| + \ln\left(x^2+1\right) + 3\tan^{-1}x + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{x^2+5x+2}{(x+1)(x^2+1)}\,dx = \int\left(\frac{-1}{x+1}+\frac{2x+3}{x^2+1}\right)dx$$
$$= \int\left(-\frac{1}{x+1}+\frac{2x}{x^2+1}+\frac{3}{x^2+1}\right)dx$$
$$= -\ln|x+1| + \ln\left(x^2+1\right) + 3\tan^{-1}x + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{1}{x(x^2+x+1)}\,dx = \int\left(\frac{1}{x}+\frac{-x-1}{x^2+x+1}\right)dx$$

$$= \int\left(\frac{1}{x}-\frac{1}{2}\frac{2x}{x^2+x+1}-\frac{1}{2}\frac{1}{x^2+x+1}\right)dx$$

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right] + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{1}{x(x^2+x+1)}\,dx = \int\left(\frac{1}{x}+\frac{-x-1}{x^2+x+1}\right)dx$$

$$= \int\left(\frac{1}{x}-\frac{1}{2}\frac{2x}{x^2+x+1}-\frac{1}{2}\frac{1}{x^2+x+1}\right)dx$$

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right] + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{1}{x(x^2+x+1)}\,dx = \int\left(\frac{1}{x}+\frac{-x-1}{x^2+x+1}\right)dx$$

$$= \int\left(\frac{1}{x}-\frac{1}{2}\frac{2x}{x^2+x+1}-\frac{1}{2}\frac{1}{x^2+x+1}\right)dx$$

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right] + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example

$$\int \frac{1}{x(x^2+x+1)}\,dx = \int \left(\frac{1}{x} + \frac{-x-1}{x^2+x+1}\right) dx$$

$$= \int \left(\frac{1}{x} - \frac{1}{2}\frac{2x}{x^2+x+1} - \frac{1}{2}\frac{1}{x^2+x+1}\right) dx$$

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right] + C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{u^k}\,du = -\frac{1}{k-1}\frac{1}{u^{k-1}} + C$$

$$= -\frac{1}{k-1}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{u^k}\,du = -\frac{1}{k-1}\frac{1}{u^{k-1}} + C$$

$$= -\frac{1}{k-1}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{u^k}\,du = -\frac{1}{k-1}\frac{1}{u^{k-1}} + C$$

$$= -\frac{1}{k-1}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{u^k}\,du = -\frac{1}{k-1}\frac{1}{u^{k-1}} + C$$

$$= -\frac{1}{k-1}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{u^k}\,du = -\frac{1}{k-1}\frac{1}{u^{k-1}} + C$$

$$= -\frac{1}{k-1}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{u^k}\,du = -\frac{1}{k-1}\frac{1}{u^{k-1}} + C$$

$$= -\frac{1}{k-1}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{u^k}\,du = -\frac{1}{k-1}\frac{1}{u^{k-1}} + C$$

$$= -\frac{1}{k-1}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{u^k}\,du = -\frac{1}{k-1}\frac{1}{u^{k-1}} + C$$

$$= -\frac{1}{k-1}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{1}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{(t^2+a^2)^k}\,dt = \int \frac{1}{(a^2\sec^2 u)^k}\,a\sec^2 u\,du$$

$$= \frac{1}{a^{2k-1}}\int \frac{1}{\sec^{2(k-1)} u}\,du = \frac{1}{a^{2k-1}}\int \cos^{2(k-1)} u\,du = \cdots$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.

Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.

set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Reduction: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{1}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{(t^2+a^2)^k}\,dt = \int \frac{1}{(a^2\sec^2 u)^k}\,a\sec^2 u\,du$$

$$= \frac{1}{a^{2k-1}}\int \frac{1}{\sec^{2(k-1)} u}\,du = \frac{1}{a^{2k-1}}\int \cos^{2(k-1)} u\,du = \cdots$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.

Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.

set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Reduction: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{1}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{(t^2+a^2)^k}\,dt = \int \frac{1}{(a^2\sec^2 u)^k}\,a\sec^2 u\,du$$

$$= \frac{1}{a^{2k-1}}\int \frac{1}{\sec^{2(k-1)} u}\,du = \frac{1}{a^{2k-1}}\int \cos^{2(k-1)} u\,du = \cdots$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.

Set $t = x+\beta/2,\ a^2 = \gamma-\beta^2/4$.

set $a\tan u = t,\ a\sec^2 u\,du = dt,\ t^2+a^2 = a^2\sec^2 u$.

Reduction: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{1}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{(t^2+a^2)^k}\,dt = \int \frac{1}{(a^2\sec^2 u)^k}\,a\sec^2 u\,du$$

$$= \frac{1}{a^{2k-1}}\int \frac{1}{\sec^{2(k-1)} u}\,du = \frac{1}{a^{2k-1}}\int \cos^{2(k-1)} u\,du = \cdots$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.

Set $t = x+\beta/2,\ a^2 = \gamma-\beta^2/4$.

set $a\tan u = t,\ a\sec^2 u\,du = dt,\ t^2+a^2 = a^2\sec^2 u$.

Reduction: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{1}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{(t^2+a^2)^k}\,dt = \int \frac{1}{(a^2\sec^2 u)^k}\,a\sec^2 u\,du$$

$$= \frac{1}{a^{2k-1}}\int \frac{1}{\sec^{2(k-1)} u}\,du = \frac{1}{a^{2k-1}}\int \cos^{2(k-1)} u\,du = \cdots$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2 + \gamma - \beta^2/4$.
Set $t = x+\beta/2,\ a^2 = \gamma - \beta^2/4$.
set $a\tan u = t,\ a\sec^2 u\,du = dt,\ t^2+a^2 = a^2\sec^2 u$.

Reduction: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{1}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{(t^2+a^2)^k}\,dt = \int \frac{1}{(a^2\sec^2 u)^k}\,a\sec^2 u\,du$$

$$= \frac{1}{a^{2k-1}}\int \frac{1}{\sec^{2(k-1)} u}\,du = \frac{1}{a^{2k-1}}\int \cos^{2(k-1)} u\,du = \cdots$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2 + \gamma - \beta^2/4$.

Set $t = x+\beta/2$, $a^2 = \gamma - \beta^2/4$.

set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Reduction: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{1}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{(t^2+a^2)^k}\,dt = \int \frac{1}{(a^2\sec^2 u)^k}\,a\sec^2 u\,du$$

$$= \frac{1}{a^{2k-1}}\int \frac{1}{\sec^{2(k-1)} u}\,du = \frac{1}{a^{2k-1}}\int \cos^{2(k-1)} u\,du = \cdots$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.

Set $t = x+\beta/2,\ a^2 = \gamma-\beta^2/4$.

set $a\tan u = t,\ a\sec^2 u\,du = dt,\ t^2+a^2 = a^2\sec^2 u$.

Reduction: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{1}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{(t^2+a^2)^k}\,dt = \int \frac{1}{(a^2\sec^2 u)^k}a\sec^2 u\,du$$

$$= \frac{1}{a^{2k-1}}\int \frac{1}{\sec^{2(k-1)} u}\,du = \frac{1}{a^{2k-1}}\int \cos^{2(k-1)} u\,du = \cdots$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.

Set $t = x+\beta/2,\ a^2 = \gamma-\beta^2/4$.

set $a\tan u = t,\ a\sec^2 u\,du = dt,\ t^2+a^2 = a^2\sec^2 u$.

Reduction: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

where $c = \frac{C-\frac{B}{2}\beta}{a^{2k-1}}$, $t = x+\beta/2$, $a^2 = \gamma - \beta^2/4$, $a\tan u = t$.

Example

$$\int \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}\,dx = \int \left(\frac{2}{x+1} + \frac{x}{x^2+4} - \frac{1}{(x^2+4)^2}\right) dx$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{8}\int \cos^2 u\,du$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}\left(u + \sin u\cos u\right) + C$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right) + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

where $c = \frac{C-\frac{B}{2}\beta}{a^{2k-1}}$, $t = x+\beta/2$, $a^2 = \gamma - \beta^2/4$, $a\tan u = t$.

Example

$$\int \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}\,dx = \int\left(\frac{2}{x+1} + \frac{x}{x^2+4} - \frac{1}{(x^2+4)^2}\right)dx$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{8}\int \cos^2 u\,du$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}(u+\sin u\cos u) + C$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right) + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

where $c = \dfrac{C-\frac{B}{2}\beta}{a^{2k-1}}$, $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$, $a\tan u = t$.

Example

$$\int \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}\,dx = \int\left(\frac{2}{x+1}+\frac{x}{x^2+4}-\frac{1}{(x^2+4)^2}\right)dx$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{8}\int \cos^2 u\,du$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}(u+\sin u\cos u) + C$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}\left(\tan^{-1}\frac{x}{2}+\frac{2x}{x^2+4}\right) + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

where $c = \frac{C-\frac{B}{2}\beta}{a^{2k-1}}$, $t = x+\beta/2$, $a^2 = \gamma - \beta^2/4$, $a\tan u = t$.

Example

$$\int \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}\,dx = \int\left(\frac{2}{x+1} + \frac{x}{x^2+4} - \frac{1}{(x^2+4)^2}\right)dx$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{8}\int \cos^2 u\,du$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}(u + \sin u\cos u) + C$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right) + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

where $c = \dfrac{C-\frac{B}{2}\beta}{a^{2k-1}}$, $t = x+\beta/2$, $a^2 = \gamma - \beta^2/4$, $a\tan u = t$.

Example

$$\int \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}\,dx = \int \left(\frac{2}{x+1} + \frac{x}{x^2+4} - \frac{1}{(x^2+4)^2}\right) dx$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{8}\int \cos^2 u\,du$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}(u+\sin u\cos u) + C$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right) + C$$

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

where $c = \frac{C-\frac{B}{2}\beta}{a^{2k-1}}$, $t = x+\beta/2$, $a^2 = \gamma - \beta^2/4$, $a\tan u = t$.

Example

$$\int \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}\,dx = \int\left(\frac{2}{x+1} + \frac{x}{x^2+4} - \frac{1}{(x^2+4)^2}\right)dx$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{8}\int \cos^2 u\,du$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}(u+\sin u\cos u) + C$$

$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right) + C$$

Outline

- Partial Fraction Decomposition
  - Rational Function
  - Partial Fraction Decomposition
- Integrals of Partial Fractions
  - Partial Fractions