Lecture 10Section 8.5 Rational Functions; Partial Fractions

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1 Partial Fraction Decomposition

1.1 Rational Function

Rational Function

- *Rational function*: $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
  - Yes: $\dfrac{2x}{x^2 - x - 2}, \quad \dfrac{3x^4 - 20x^2 + 17}{x^3 + 2x^2 - 7}$
  - No: $\dfrac{1}{\sqrt{x}}, \quad \dfrac{x^2 + 1}{\ln x}$
- If degree(P) $\geq$ degree(Q), then, by division, $$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}$$ where $p(x)$ is a polynomial and $\frac{r(x)}{Q(x)}$ is a *proper rational function* (i.e., degree(r) $<$ degree(Q)).
- $\dfrac{x^2}{x^2 - 2x - 3} = 1 + \dfrac{2x + 3}{x^2 - 2x - 3}$
- $\dfrac{x^3}{x^2 - 2x + 1} = x + 2 + \dfrac{3x - 2}{x^2 - 2x + 1}$

1.2 Partial Fraction Decomposition

Partial Fraction Decomposition
Partial Fraction Decomposition

- Let the denominator $Q(x)$ factor as $$Q(x) = a \prod (x - \alpha_i)^{n_i} \prod (x^2 + \beta_j x + \gamma_j)^{n_j}$$ where the quadratic factors $x^2 + \beta_j x + \gamma_j$ are *irreducible* (i.e., $\beta_j^2 - 4\gamma_j < 0$, they have complex zeros).
- *Proper rational function* $R(x) = \frac{P(x)}{Q(x)}$ can be written as a sum of *partial fractions* of the form: $$\frac{A}{(x - \alpha)^k}, \quad \frac{Bx + C}{(x^2 + \beta x + \gamma)^k} \quad :$$
  - Each power $(x - \alpha)^n$ of a linear factor $x - \alpha$ contributes: $$\frac{A_n}{(x - \alpha)^n} + \cdots \frac{A_2}{(x - \alpha$$

– Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes: $\dfrac{B_n x + C_n}{(x^2+\beta x+\gamma)^n} + \cdots \dfrac{B_2 x + C_2}{(x^2+\beta x+\gamma)^2} + \dfrac{B_1 x + C_1}{x^2+\beta x+\gamma}$

Partial Fraction Decomposition: Example

- Each power $(x-\alpha)^n$ of a linear factor $x-\alpha$ contributes: $\dfrac{A_n}{(x-\alpha)^n} + \cdots \dfrac{A_2}{(x-\alpha)^2} +$
- Each power $(x^2+\beta x+\gamma)^n$ of an irreducible quadratic factor $x^2+\beta x+\gamma$ contributes: $\dfrac{B_n x + C_n}{(x^2+\beta x+\gamma)^n} + \cdots \dfrac{B_2 x + C_2}{(x^2+\beta x+\gamma)^2} + \dfrac{B_1 x + C_1}{x^2+\beta x+\gamma}$

Examples 1.

$$\frac{1}{x^3(x^2+1)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{Dx+E}{x^2+1} = \frac{1}{x^3} - \frac{1}{x} + \frac{x}{x^2+1}$$

$$\begin{aligned}
&1 = (C+D)x^4 + (B+E)x^3 + (A+C)x^2 + Bx + A \\
&A = 1, \quad B = 0, \quad A + C = 0, \quad B + E = 0, \quad C + D = 0 \\
&A = 1, \quad B = 0, \quad C = -1, \quad E = 0, \quad D = 1
\end{aligned}$$

$$\frac{x^2}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+1)^2} + \frac{Dx+E}{x^2+1}$$

$$\begin{aligned}
\Rightarrow \quad & x^2 = (A+D)x^4 + (-D+E)x^3 + (2A+B+D-E)x^2 \\
& \quad + (-B+C-D+E)x + (A-C-E) \\
\Rightarrow \quad & A + D = 0, \quad -D + E = 0, \quad 2A + B + D - E = 1, \\
& -B + C - D + E = 0, \quad A - C - E = 0 \quad \Rightarrow \text{Finish it}
\end{aligned}$$

$$\frac{x^3}{(x^2+2x+2)^2} = \frac{Ax+B}{(x^2+2x+2)^2} + \frac{Cx+D}{x^2+2x+2}$$

$$\begin{aligned}
\Rightarrow \quad & x^3 = Cx^3 + (2C+D)x^2 + (A+2C+2D)x + (B+2D) \\
\Rightarrow \quad & C = 1, \quad 2C + D = 0, \quad A + 2C + 2D = 0, \quad B + 2D = 0 \\
\Rightarrow & \text{ Finish it}
\end{aligned}$$

2 Integrals of Partial Fractions

2.1 Partial Fractions

Integrals of Partial Fractions

Partial Fractions: $\frac{A}{(x-\alpha)^k}$

$$\int \frac{A}{x-\alpha}\,dx = A\ln|x-\alpha| + C$$
$$\int \frac{A}{(x-\alpha)^k}\,dx = -\frac{A}{k-1}\,\frac{1}{(x-\alpha)^{k-1}} + C$$

Examples 2.

$$\begin{aligned}\int \frac{6}{x^3-5x^2+6x}\,dx &= \int \frac{6}{x(x-2)(x-3)}\,dx \\ &= \int \left(\frac{1}{x} - \frac{3}{x-2} + \frac{2}{x-3}\right)dx \\ &= \ln|x| - 3\ln|x-2| + 2\ln|x-3| + C\end{aligned}$$

$$\begin{aligned}\int \frac{1}{x^3-2x^2}\,dx &= \int \frac{1}{x^2(x-2)}\,dx \\ &= \int \left(\frac{-1/2}{x^2} + \frac{-1/4}{x} + \frac{1/4}{x-2}\right)dx \\ &= \frac{1}{4}\left(\frac{2}{x} - \ln|x| + \ln|x-2|\right) + C\end{aligned}$$

$$\begin{aligned}\int \frac{x^2}{(x^2-9)^2}\,dx &= \int \frac{x^2}{(x-3)^2(x+3)^2}\,dx \\ &= \int \left(\frac{1/4}{(x-3)^2} + \frac{1/12}{x-3} + \frac{1/4}{(x+3)^2} + \frac{-1/12}{x+3}\right)dx \\ &= \frac{1}{12}\left(-\frac{3}{x-3} + \ln|x-3| - \frac{3}{x+3} - \ln|x+3|\right) + C\end{aligned}$$

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| \quad + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Proof.

$$\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|x^2+\beta x+\gamma\right| + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

$$\int \frac{1}{x^2+\beta x+\gamma}\,dx = \int \frac{1}{t^2+a^2}\,dt = \int \frac{1}{a^2\sec^2 u} a\sec^2 u\,du$$
$$= \frac{1}{a}\int du = \frac{1}{a}u + C = \frac{1}{a}\tan^{-1}\frac{t}{a} = \frac{1}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.
Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.
set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.

Integrals of Partial Fractions: Examples

Partial Fractions: $\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2}\frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{C-\frac{B}{2}\beta}{x^2+\beta x+\gamma}$

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx = \frac{B}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{C-\frac{B}{2}\beta}{\sqrt{\gamma-\frac{\beta^2}{4}}}\tan^{-1}\frac{x+\frac{\beta}{2}}{\sqrt{\gamma-\frac{\beta^2}{4}}}$$

Example 3.

$$\int \frac{x^2}{(x+1)(x^2+4)}\,dx = \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x-4}{x^2+4}\right)dx$$
$$= \frac{1}{5}\int\left(\frac{1}{x+1}+\frac{4x}{x^2+4}-\frac{4}{x^2+4}\right)dx$$
$$= \frac{1}{5}\left(\ln|x+1| + 2\ln\left(x^2+4\right) - 2\tan^{-1}\frac{x}{2}\right)+C$$

$$\int \frac{x^2+5x+2}{(x+1)(x^2+1)}\,dx = \int\left(\frac{-1}{x+1}+\frac{2x+3}{x^2+1}\right)dx$$
$$= \int\left(-\frac{1}{x+1}+\frac{2x}{x^2+1}+\frac{3}{x^2+1}\right)dx$$
$$= -\ln|x+1| + \ln\left(x^2+1\right) + 3\tan^{-1}x + C$$

$$\int \frac{1}{x(x^2+x+1)}\,dx = \int\left(\frac{1}{x}+\frac{-x-1}{x^2+x+1}\right)dx$$
$$= \int\left(\frac{1}{x}-\frac{1}{2}\frac{2x}{x^2+x+1}-\frac{1}{2}\frac{1}{x^2+x+1}\right)dx$$
$$= \ln|x| - \frac{1}{2}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]+C$$

Integrals of Partial Fractions

Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

Proof.

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{u^k}\,du = -\frac{1}{k-1}\frac{1}{u^{k-1}} + C$$
$$= -\frac{1}{k-1}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + C$$

Set $u = x^2+\beta x+\gamma$, $du = 2x+\beta\,dx$.

$$\int \frac{1}{(x^2+\beta x+\gamma)^k}\,dx = \int \frac{1}{(t^2+a^2)^k}\,dt = \int \frac{1}{(a^2\sec^2 u)^k}\,a\sec^2 u\,du$$
$$= \frac{1}{a^{2k-1}}\int \frac{1}{\sec^{2(k-1)} u}\,du = \frac{1}{a^{2k-1}}\int \cos^{2(k-1)} u\,du = \cdots$$

Note $x^2+\beta x+\gamma = (x+\beta/2)^2+\gamma-\beta^2/4$.
Set $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$.
set $a\tan u = t$, $a\sec^2 u\,du = dt$, $t^2+a^2 = a^2\sec^2 u$.
Reduction: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

Integrals of Partial Fractions: Examples
Partial Fractions: $\frac{Bx+C}{(x^2+\beta x+\gamma)^k} = \frac{B}{2}\frac{2x+\beta}{(x^2+\beta x+\gamma)^k} + \frac{C-\frac{B}{2}\beta}{(x^2+\beta x+\gamma)^k}$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = -\frac{B}{2(k-1)}\frac{1}{(x^2+\beta x+\gamma)^{k-1}} + c\int \cos^{2(k-1)} u\,du$$

where $c = \frac{C-\frac{B}{2}\beta}{a^{2k-1}}$, $t = x+\beta/2$, $a^2 = \gamma-\beta^2/4$, $a\tan u = t$.

Example 4.

$$\int \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}\,dx = \int \left(\frac{2}{x+1} + \frac{x}{x^2+4} - \frac{1}{(x^2+4)^2}\right)dx$$
$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{8}\int \cos^2 u\,du$$
$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}(u+\sin u\cos u) + C$$
$$= 2\ln|x+1| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right) + C$$

Outline

Contents