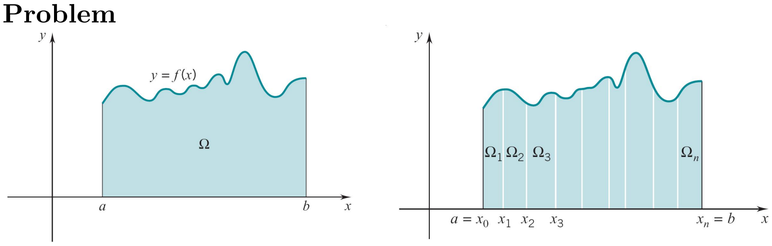
Lecture 11 Section 8.7 Numerical Integration

Jiwen He

1 Riemann Sums

1.1 Area Problem

Area Problem

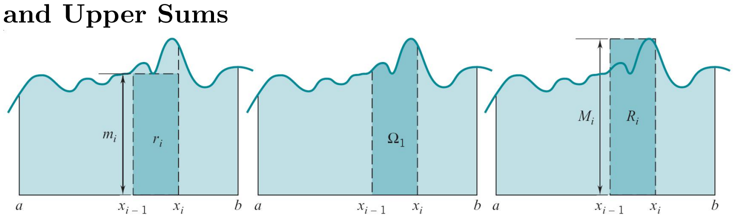


Partition of $[a, b]$
Take a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$. Then P splits up the interval $[a, b]$ into a finite number of subintervals $[x_0, x_1]$, $\cdots$, $[x_{n-1}, x_n]$ with $a = x_0 < x_1 < \cdots < x_n = b$. We have $[a, b] = [x_0, x_1] \cup \cdots \cup [x_{i-1}, x_i] \cup \cdots \cup [x_{n-1}, x_n]$

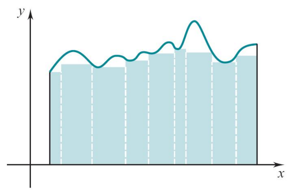
Remark
This breaks up the region Ω into n subregions Ω_1,$\cdots$,Ω_n: $\Omega = \Omega_1 \cup \cdots \cup \Omega_i \cup \cdots \cup \Omega_n$
We can *estimate the total area of* Ω *by estimating the area of each subregion* Ω_i *and adding up the results.*

1.2 Lower and Upper Sums

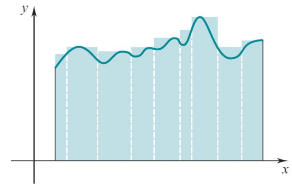
Lower and Upper Sums



Let $\Delta x_i = x_i - x_{i-1}$, $m_i = \min_{x \in [x_{i-1}, x_i]} f(x)$, $M_i = \max_{x \in [x_{i-1}, x_i]} f(x)$ $m_i \Delta x_i = \text{area of } r_i \leq \text{area of } \Omega_i \leq$



area of shaded region is a lower sum for f

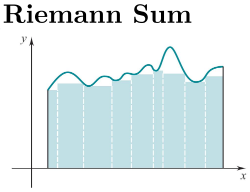


area of shaded region is an upper sum for f

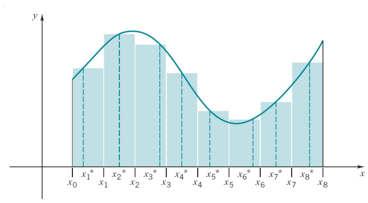
$$L_f(P) := \sum_{i=1}^{n} m_i \Delta x_i \le I = \int_a^b f(x)\,dx \le \sum_{i=1}^{n} M_i \Delta x_i =: U_f(P)$$

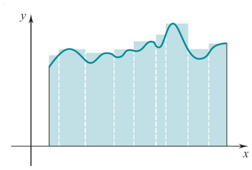
1.3 Riemann Sum

Riemann Sum



area of shaded region is a lower sum for f





area of shaded region is an upper sum for f

Riemann Sum

$$S^*(P) = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

where x_i^* *is any point picked* in $[x_{i-1}, x_i]$ for $i = 1, \cdots, n$. We have

$$L_f(P) := \sum_{i=1}^{n} m_i \Delta x_i \le S^*(P) := \sum_{i=1}^{n} f(x_i^*)\Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i =: U_f(P)$$

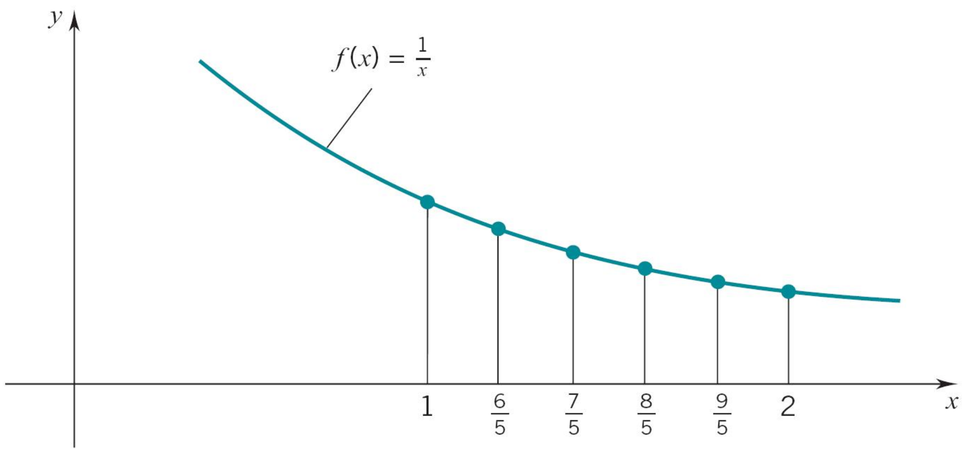
Limit of Riemann Sums

$$\int_a^b f(x)\,dx = \lim_{|P|\to 0} [f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n]$$

2 Rectangle Approximations

2.1 Regular Partition

Regular Partition



Problem

Find the approximate value of $$\ln 2 = \int_1^2 \frac{dx}{x}$$

using only the values of $f(x) = \frac{1}{x}$ at $$1, \quad \frac{6}{5}, \quad \frac{7}{5}, \quad \frac{8}{5}, \quad \frac{9}{5}, 2.$$

Numerical Integration

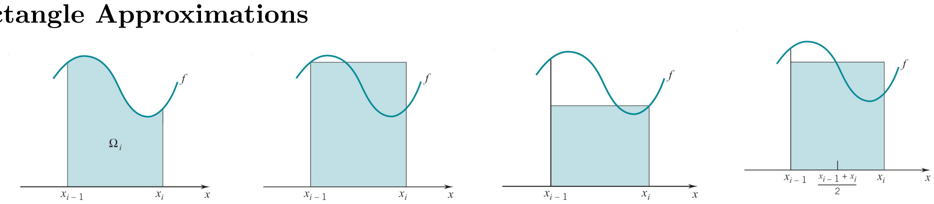
Approximate $\int_a^b f(x)\,dx$ using only values of f at $n+1$ equally-spaced points between a and b.

Regular Partition of $[a, b]$

Let $x_i = a + i\Delta x$, $i = 0, 1, \cdots, n$, where $\Delta x = \frac{b-a}{n}$. Then $$[a, b] = [x_0, x_1] \cup \cdots \cup [x_{i-1}, x_i] \cup \cdots \cup$$

2.2 Approximations

Rectangle Approximations



$\int_{x_{i-1}}^{x_i} f(x)\,dx$ $\qquad f(x_{i-1})\Delta x$ $\qquad f(x_i)\Delta x$ $\qquad f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$

Rectangle Approximations of $\int_{x_{i-1}}^{x_i} f(x)\,dx$

$f(x_{i-1})\Delta x$,	$f(x_i)\Delta x$,	$f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$
left-endpoints	*right-endpoints*	*midpoints*

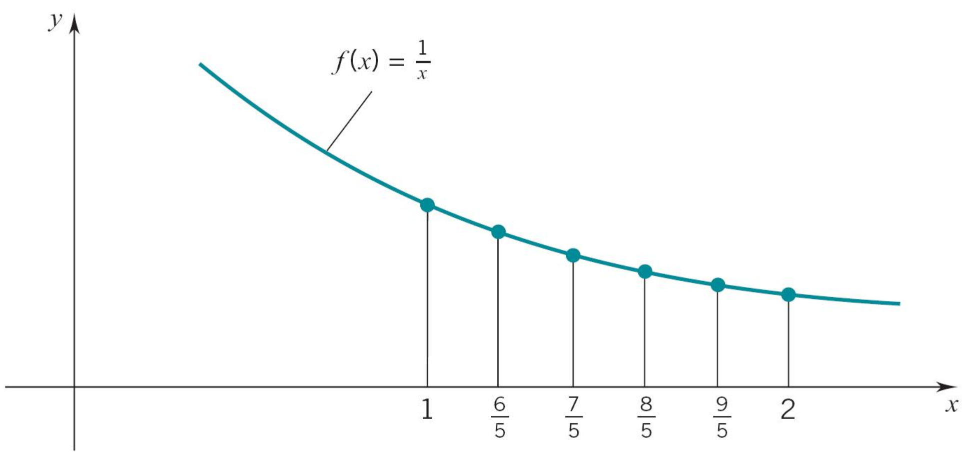
Rectangle Approximations of $\int_a^b f(x)\,dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)\,dx$

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x, \quad \sum_{i=1}^{n} f(x_i)\Delta x, \quad \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

left-endpoints *right-endpoints* *midpoints*

Rectangle Rules
Rectangle Rule for Approximating $\int_a^b f(x)\,dx$

- Left-endpoint rule: $$L_n = \frac{b-a}{n}\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right]$$
- Right-endpoint rule: $$R_n = \frac{b-a}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$$
- Midpoint rule: $$M_n = \frac{b-a}{n}\left[f\left(\frac{x_0+x_1}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$$



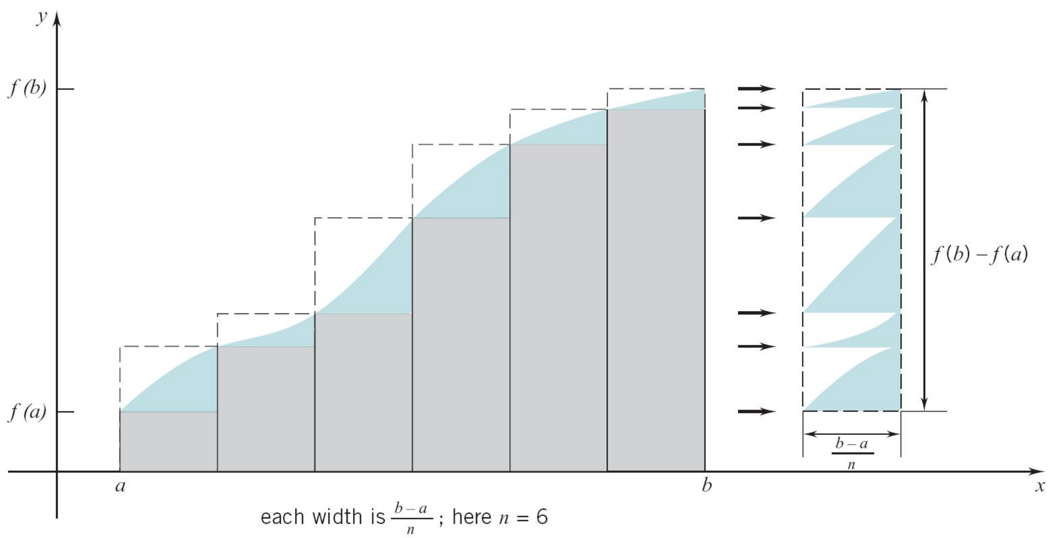
Problem
Find the approximate value of $\ln 2 = \int_1^2 \frac{dx}{x}$ using only the values of $f(x) = \frac{1}{x}$ at $1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2$.

Approximating $\ln 2 = 0.69314718\cdots$

$$L_5 = \frac{1}{5}\left(1 + \frac{5}{6} + \frac{5}{7} + \frac{5}{8} + \frac{5}{9}\right) \approx 0.7456$$
$$R_5 = \frac{1}{5}\left(\frac{5}{6} + \frac{5}{7} + \frac{5}{8} + \frac{5}{9} + \frac{1}{2}\right) \approx 0.6456$$
$$M_5 = \frac{1}{5}\left(\frac{10}{11} + \frac{10}{13} + \frac{10}{15} + \frac{10}{17} + \frac{10}{19}\right) \approx 0.6919$$

2.3 Error Estimates

Error Estimates



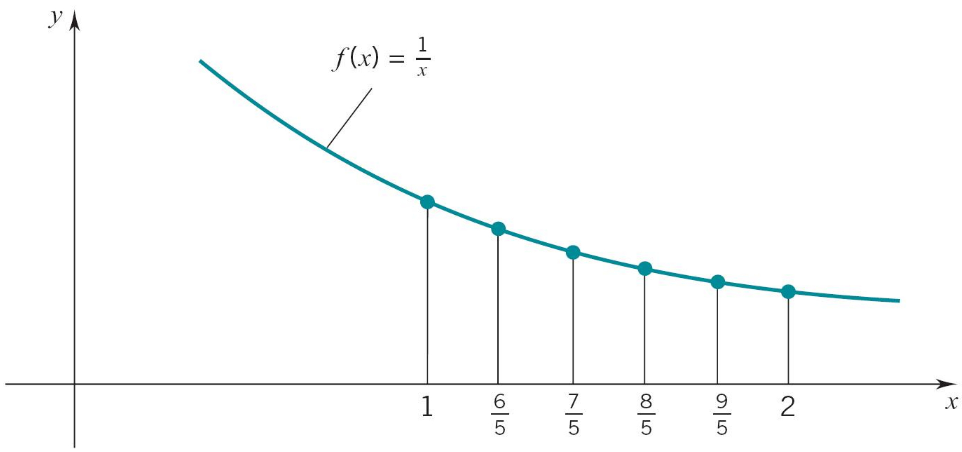
$$\left|\int_a^b f(x)\,dx - L_n\right| \le \left(\frac{b-a}{n}\right)[f(b)-f(a)]$$

Error Estimates

- Left-endpoint rule: $$E_n^L = \int_a^b f(x)\,dx - L_n = \frac{1}{2}\frac{(b-a)^2}{n}f'(c) = O(\Delta x)$$
- Right-endpoint rule: $$E_n^R = \int_a^b f(x)\,dx - R_n = \frac{1}{2}\frac{(b-a)^2}{n}f'(c) = O(\Delta x)$$
- Midpoint rule: $$E_n^M = \int_a^b f(x)\,dx - M_n = \frac{1}{24}\frac{(b-a)^3}{n^2}f''(c) = O((\Delta x)^2)$$

Error Estimates: Example
Error Estimates

- Left-endpoint rule: $$E_n^L = \int_a^b f(x)\,dx - L_n = \frac{1}{2}\frac{(b-a)^2}{n}f'(c) = O(\Delta x)$$
- Right-endpoint rule: $$E_n^R = \int_a^b f(x)\,dx - R_n = \frac{1}{2}\frac{(b-a)^2}{n}f'(c) = O(\Delta x)$$
- Midpoint rule: $$E_n^M = \int_a^b f(x)\,dx - M_n = \frac{1}{24}\frac{(b-a)^3}{n^2}f''(c) = O((\Delta x)^2)$$



Approximating $\ln 2 = 0.69314718\cdots$

Note that $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$.

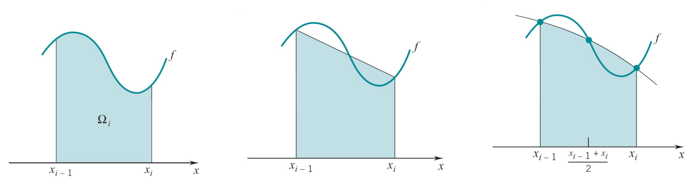
$$|E_n^L| = |E_n^R| \le \frac{1}{2}\frac{(2-1)^2}{5}\max_{c\in[1,2]}|f'(c)| = \frac{1}{10}$$

$$|E_n^M| \le \frac{1}{24}\frac{(2-1)^3}{5^2}\max_{c\in[1,2]}|f''(c)| = \frac{1}{300}$$

3 Trapeziodal and Parabolic Approximations

3.1 Approximations

Trapeziodal and Parabolic Approximations



$\int_{x_{i-1}}^{x_i} f(x)\,dx$ Trapeziodal Parabolic

Approximations of $\int_{x_{i-1}}^{x_i} f(x)\,dx$

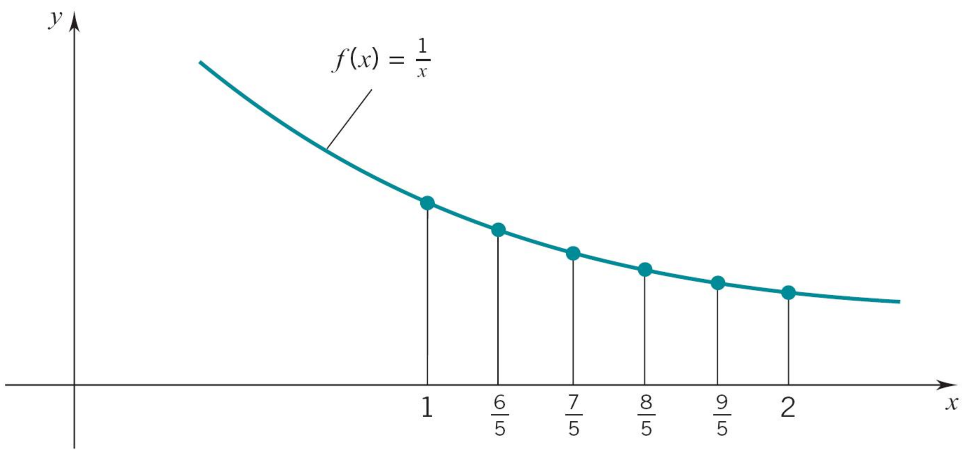
$$\underbrace{\frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]\Delta x}_{\textit{trapeziodal}}, \quad \underbrace{\frac{1}{6}\left[f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)\right]\Delta x}_{\textit{Parabolic}}$$

Approximations of $\int_a^b f(x)\,dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)\,dx$

$$\underbrace{\sum_{i=1}^n \frac{\Delta x}{2}\left[f(x_{i-1}) + f(x_i)\right]}_{\textit{Trapeziodal}}, \underbrace{\sum_{i=1}^n \frac{\Delta x}{6}\left[f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)\right]}_{\textit{Parabolic}}$$

Trapeziodal and Simpson's Rules
Trapeziodal and Simpson's Rules for Approximating $\int_a^b f(x)\,dx$

- Trapeziodal Rule: $$T_n = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$
- Simpson's Rule (*Parabolic*): $$S_n = \frac{b-a}{6n}\left\{f(x_0) + f(x_n) + 2\left[f(x_1) + \cdots + 2f(x_{n-1})\right] + 4\left[f\left(\frac{x_0+x_1}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]\right\}$$



Problem
Find the approximate value of $\ln 2 = \int_1^2 \frac{dx}{x}$ using only the values of $f(x) = \frac{1}{x}$ at $1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2$.

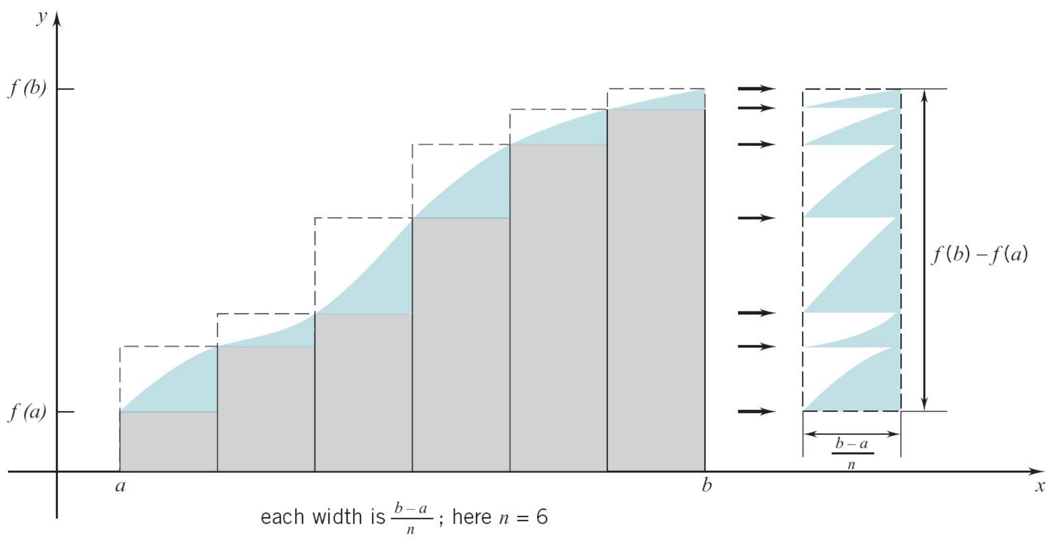
Approximating $\ln 2 = 0.69314718\cdots$

$$T_5 = \frac{1}{10}\left(1 + \frac{10}{6} + \frac{10}{7} + \frac{10}{8} + \frac{10}{9} + \frac{1}{2}\right) \approx 0.6956$$

$$S_5 = \frac{1}{30}\left\{1 + \frac{1}{2} + 2\left(\frac{5}{6} + \frac{5}{7} + \frac{5}{8} + \frac{5}{9}\right) + 4\left[\frac{10}{11} + \frac{10}{13} + \frac{10}{15} + \frac{10}{17} + \frac{10}{19}\right]\right\} \approx 0.6932$$

3.2 Error Estimates

Error Estimates



$$\left|\int_a^b f(x)\,dx - T_n\right| \leq \frac{1}{2}\left(\frac{b-a}{n}\right)[f(b)-f(a)]$$

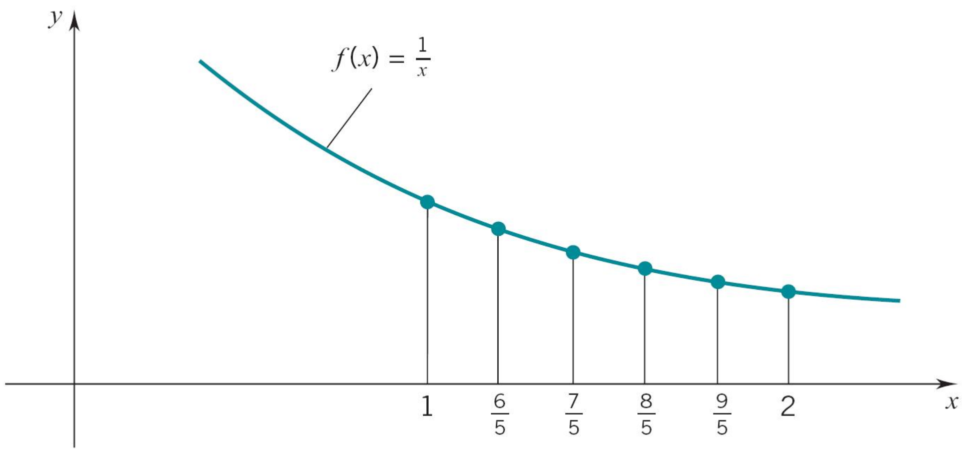
Error Estimates

- Trapeziodal Rule: $E_n^T = \int_a^b f(x)\,dx - T_n = -\frac{1}{12}\frac{(b-a)^3}{n^2}f''(c) = O((\Delta x)^2)$
- Simpson's Rule: $E_n^S = \int_a^b f(x)\,dx - S_n = -\frac{1}{2880}\frac{(b-a)^5}{n^4}f^{(4)}(c) = O((\Delta x)^4)$

Error Estimates: Example

Error Estimates

- Trapeziodal Rule: $E_n^T = \int_a^b f(x)\,dx - T_n = -\frac{1}{12}\frac{(b-a)^3}{n^2}f''(c) = O((\Delta x)^2)$
- Simpson's Rule: $E_n^S = \int_a^b f(x)\,dx - S_n = -\frac{1}{2880}\frac{(b-a)^5}{n^4}f^{(4)}(c) = O((\Delta x)^4)$



Approximating $\ln 2 = 0.69314718\cdots$

Note that $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f^{(3)}(x) = -\frac{6}{x^4}$, $f^{(4)}(x) = \frac{24}{x^5}$.

$$|E_n^T| \leq \frac{1}{12}\frac{(2-1)^3}{5^2}\max_{c\in[1,2]}|f''(c)| = \frac{1}{150}$$

$$|E_n^M| \leq \frac{1}{2880}\frac{(2-1)^5}{5^4}\max_{c\in[1,2]}|f^{(4)}(c)| = \cdots$$

Outline

Contents