# Lecture 12 <br> <br> Section 9.3 Polar Coordinates <br> <br> Section 9.3 Polar Coordinates Section 9.4 Graphing in Polar Coordinates 

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spiral of Archimedes

## Polar Coordinate System



## Frame of Reference <br> In the polar coordinate system, the frame of reference is a point $O$ that we call the pole and a ray that emanates from it that we call the polar axis.

## Polar Coordinate System



## Frame of Reference

In the polar coordinate system, the frame of reference is a point $O$ that we call the pole and a ray that emanates from it that we call the polar axis.

## Polar Coordinate System



The purpose of the polar coordinates is to represent curves that have symmetry about a point or spiral about a point.

## Frame of Reference

In the polar coordinate system, the frame of reference is a point $O$ that we call the pole and a ray that emanates from it that we call the polar axis.

## Polar Coordinates



## Definition

A point is given polar coordinates $[r, \theta]$ iff it lies at a distance $|r|$ from the pole a long the ray $\theta$, if $r \geq 0, \quad$ and $\quad$ along the ray $\theta+\pi$, if $r<0$.

## Points in Polar Coordinates



## Points in Polar Coordinates

- $O=[0, \theta]$ for all $\theta$.



## Points in Polar Coordinates



## Points in Polar Coordinates

- $O=[0, \theta]$ for all $\theta$.
- $[r, \theta]=[r, \theta+2 n \pi]$ for all integers $n$.



## Points in Polar Coordinates



## Points in Polar Coordinates

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- $[r,-\theta]=[r, \theta+\pi]$.


## Points in Polar Coordinates



## Points in Polar Coordinates

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- $[r, \theta]=[r, \theta+2 n \pi]$ for all integers $n$.
- $[r,-\theta]=[r, \theta+\pi]$.


## Relation to Rectangular Coordinates




Relation to Rectangular Coordinates

$$
x=r \cos \theta, y=r \sin \theta
$$

## Relation to Rectangular Coordinates




Relation to Rectangular Coordinates

- $x=r \cos \theta, y=r \sin \theta$.



## Relation to Rectangular Coordinates




Relation to Rectangular Coordinates
$x=r \cos \theta, y=r \sin \theta . \quad \Rightarrow \quad x^{2}+y^{2}=r^{2}, \quad \tan \theta=\frac{y}{x}$

## Relation to Rectangular Coordinates




Relation to Rectangular Coordinates

- $x=r \cos \theta, y=r \sin \theta . \quad \Rightarrow \quad x^{2}+y^{2}=r^{2}, \quad \tan \theta=\frac{y}{x}$
- $r=\sqrt{x^{2}+y^{2}}, \theta=\tan ^{-1} \frac{y}{x}$.


## Circles in Polar Coordinates


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circles in Polar Coordinates

In rectangular coordinates
In polar coordinates
$\square$

## Circles in Polar Coordinates


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circles in Polar Coordinates

In rectangular coordinates

$$
x^{2}+y^{2}=a^{2}
$$

In polar coordinates
$c=$

## Circles in Polar Coordinates


$r=2$

$r=-2$

$r=4 \sin \theta$

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## Circles in Polar Coordinates

In rectangular coordinates

$$
x^{2}+y^{2}=a^{2}
$$

In polar coordinates
$r=a$

$$
x^{2}+y^{2}=a^{2} \quad \Rightarrow \quad r^{2}=a^{2}
$$

## Circles in Polar Coordinates


$r=2$

$r=-2$

$r=4 \sin \theta$

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## Circles in Polar Coordinates

In rectangular coordinates

$$
x^{2}+y^{2}=a^{2}
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In polar coordinates
$r=a$


$$
x^{2}+y^{2}=a^{2} \quad \Rightarrow \quad r^{2}=a^{2}
$$

## Circles in Polar Coordinates


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circles in Polar Coordinates

In rectangular coordinates

$$
\begin{aligned}
x^{2}+y^{2} & =a^{2} \\
x^{2}+(y-a)^{2} & =a^{2}
\end{aligned}
$$

In polar coordinates
$r=a$
$r$


## Circles in Polar Coordinates


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circles in Polar Coordinates

In rectangular coordinates

$$
\begin{array}{rlrl}
x^{2}+y^{2} & =a^{2} & r & =a \\
x^{2}+(y-a)^{2} & =a^{2} & r & =2
\end{array}
$$

In polar coordinates
$r=2 a \sin \theta$

$$
x^{2}+(y-a)^{2}=a^{2} \Rightarrow x^{2}+y^{2}=2 a y \quad \Rightarrow \quad r^{2}=2 a r \sin \theta \text { 拈 }
$$

## Circles in Polar Coordinates


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circles in Polar Coordinates

In rectangular coordinates

$$
\begin{aligned}
x^{2}+y^{2} & =a^{2} & & r=a \\
x^{2}+(y-a)^{2} & =a^{2} & & r=2 a \sin \theta
\end{aligned}
$$

In polar coordinates

$$
x^{2}+(y-a)^{2}=a^{2} \Rightarrow x^{2}+y^{2}=2 a y \quad \Rightarrow \quad r^{2}=2 a r \sin \theta \text { 拈 }
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## Circles in Polar Coordinates


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## Circles in Polar Coordinates

In rectangular coordinates

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\begin{aligned}
x^{2}+y^{2} & =a^{2} & & r=a \\
x^{2}+(y-a)^{2} & =a^{2} & & r=2 a \sin \theta \\
(x-a)^{2}+y^{2} & =a^{2} & & r 2 a \cos \theta
\end{aligned}
$$

In polar coordinates

## Circles in Polar Coordinates


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circles in Polar Coordinates

In rectangular coordinates

$$
\begin{aligned}
x^{2}+y^{2} & =a^{2} & & r=a \\
x^{2}+(y-a)^{2} & =a^{2} & & r=2 a \sin \theta \\
(x-a)^{2}+y^{2} & =a^{2} & & r=2 a \cos \theta
\end{aligned}
$$

In polar coordinates

$$
\begin{equation*}
(x-a)^{2}+y^{2}=a^{2} \quad \Rightarrow \quad x^{2}+y^{2}=2 a x \quad \Rightarrow \quad r^{2}=2 a r \cos \theta \tag{杵}
\end{equation*}
$$

## Circles in Polar Coordinates


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circles in Polar Coordinates

In rectangular coordinates

$$
\begin{aligned}
x^{2}+y^{2} & =a^{2} & & r=a \\
x^{2}+(y-a)^{2} & =a^{2} & & r=2 a \sin \theta \\
(x-a)^{2}+y^{2} & =a^{2} & & r=2 a \cos \theta
\end{aligned}
$$

$$
\begin{equation*}
(x-a)^{2}+y^{2}=a^{2} \quad \Rightarrow \quad x^{2}+y^{2}=2 a x \quad \Rightarrow \quad r^{2}=2 a r \cos \theta \tag{杵}
\end{equation*}
$$

## Lines in Polar Coordinates


$\theta=\frac{1}{4} \pi$

$\theta=\frac{5}{4} \pi$

$r=-2 \sec \theta$

$r=2 \csc \theta$

## Lines in Polar Coordinates

In rectangular coordinates

In polar coordinates
$y=m x$
$\qquad$

## Lines in Polar Coordinates


$\theta=\frac{1}{4} \pi$

$\theta=\frac{5}{4} \pi$

$r=-2 \sec \theta$

$r=2 \csc \theta$

## Lines in Polar Coordinates

In rectangular coordinates

$$
y=m x
$$

## Lines in Polar Coordinates


$\theta=\frac{1}{4} \pi$

$\theta=\frac{5}{4} \pi$

$r=-2 \sec \theta$

$r=2 \csc \theta$

## Lines in Polar Coordinates

In rectangular coordinates

$$
y=m x
$$

In polar coordinates
$\theta=\alpha$ with $\alpha=\tan ^{-1} m$

$$
y=m x \Rightarrow \frac{y}{x}=m \Rightarrow \tan \theta=m
$$


$\theta=\frac{1}{4} \pi$

$\theta=\frac{5}{4} \pi$

$r=-2 \sec \theta$

$r=2 \csc \theta$

## Lines in Polar Coordinates

In rectangular coordinates

$$
y=m x
$$

$$
x=a
$$

In polar coordinates
$\theta=\alpha$ with $\alpha=\tan ^{-1} m$

$$
y=m x \Rightarrow \frac{y}{x}=m \Rightarrow \tan \theta=m
$$

## Lines in Polar Coordinates


$\theta=\frac{1}{4} \pi$

$\theta=\frac{5}{4} \pi$

$r=-2 \sec \theta$

$r=2 \csc \theta$

## Lines in Polar Coordinates

In rectangular coordinates

$$
\begin{array}{r}
y=m x \\
x=a
\end{array}
$$

In polar coordinates
$\theta=\alpha$ with $\alpha=\tan ^{-1} m$n
$r=a \sec \theta$
$\square$
$\square$

## Lines in Polar Coordinates


$\theta=\frac{1}{4} \pi$

$\theta=\frac{5}{4} \pi$

$r=-2 \sec \theta$

$r=2 \csc \theta$

## Lines in Polar Coordinates

In rectangular coordinates

$$
\begin{array}{r}
y=m x \\
x=a
\end{array}
$$

In polar coordinates
$\theta=\alpha$ with $\alpha=\tan ^{-1} m$
$r=a \sec \theta$

$$
x=a \Rightarrow r \cos \theta=a \quad \Rightarrow \quad r=a \sec \theta
$$

## Lines in Polar Coordinates


$\theta=\frac{1}{4} \pi$

$\theta=\frac{5}{4} \pi$

$r=-2 \sec \theta$

$r=2 \csc \theta$

## Lines in Polar Coordinates

In rectangular coordinates

$$
\begin{array}{r}
y=m x \\
x=a
\end{array}
$$

$y=a$

In polar coordinates
$\theta=\alpha$ with $\alpha=\tan ^{-1} m$
$r=a \sec \theta$

$$
x=a \Rightarrow r \cos \theta=a \quad \Rightarrow \quad r=a \sec \theta
$$

## Lines in Polar Coordinates


$\theta=\frac{1}{4} \pi$

$\theta=\frac{5}{4} \pi$

$r=-2 \sec \theta$

$r=2 \csc \theta$

## Lines in Polar Coordinates

In rectangular coordinates

$$
\begin{array}{r}
y=m x \\
x=a \\
y=a
\end{array}
$$

In polar coordinates
$\theta=\alpha$ with $\alpha=\tan ^{-1} m$
$r=a \sec \theta$
$r=a \csc \theta$

## Lines in Polar Coordinates


$\theta=\frac{1}{4} \pi$

$\theta=\frac{5}{4} \pi$

$r=-2 \sec \theta$

$r=2 \csc \theta$

## Lines in Polar Coordinates

In rectangular coordinates

$$
\begin{array}{r}
y=m x \\
x=a \\
y=a
\end{array}
$$

In polar coordinates
$\theta=\alpha$ with $\alpha=\tan ^{-1} m$
$r=a \sec \theta$
$r=a \csc \theta$

$$
y=a \Rightarrow r \sin \theta=a \Rightarrow r=a \csc \theta
$$

## Lines in Polar Coordinates


$\theta=\frac{1}{4} \pi$

$\theta=\frac{5}{4} \pi$

$r=-2 \sec \theta$

$r=2 \csc \theta$

## Lines in Polar Coordinates

In rectangular coordinates

$$
\begin{array}{r}
y=m x \\
x=a \\
y=a
\end{array}
$$

In polar coordinates
$\theta=\alpha$ with $\alpha=\tan ^{-1} m$
$r=a \sec \theta$
$r=a \csc \theta$

$$
y=a \Rightarrow r \sin \theta=a \Rightarrow r=a \csc \theta
$$

## Symmetry


symmetry about the $x$-axis

symmetry about the $y$-axis

symmetry about the origin


Lemniscate (ribbon) $r^{2}=\cos 2 \theta$
$\cos [2(-\theta)]=\cos (-2 \theta)=\cos 2 \theta$


## Symmetry


symmetry about the $x$-axis

symmetry about the $y$-axis

symmetry about the origin


Lemniscate (ribbon) $r^{2}=\cos 2 \theta$
$\cos [2(-\theta)]=\cos (-2 \theta)=\cos 2 \theta$
$\Rightarrow$ if $[r, \theta] \in$ graph, then $[r,-\theta] \in$ graph
$\Rightarrow$ symmetric about the $x$-axis.

## Symmetry


symmetry about the $x$-axis


symmetry about the $y$-axis

symmetry about the origin

$$
\begin{aligned}
& \text { Lemniscate (ribbon) } r^{2}=\cos 2 \theta \\
& \cos [2(-\theta)]=\cos (-2 \theta)=\cos 2 \theta \\
& \Rightarrow \text { if }[r, \theta] \in \text { graph, then }[r,-\theta] \in \text { graph }
\end{aligned}
$$

$\Rightarrow$ symmetric about the $x$-axis.

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$\Rightarrow$ symmetric about the $x$-axis.

## Symmetry


symmetry about the $x$-axis

symmetry about the $y$-axis

symmetry about the origin


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\begin{aligned}
& \text { Lemniscate (ribbon) } r^{2}=\cos 2 \theta \\
& \cos [2(\pi-\theta)]=\cos (2 \pi-2 \theta)=\cos 2 \theta
\end{aligned}
$$

$$
\Rightarrow \text { if }[r, \theta] \in \text { graph, then }
$$

$$
[r, \pi-\theta] \in \text { graph }
$$

$$
\Rightarrow \text { symmetric about the } y \text {-axis. }
$$

## Symmetry


symmetry about the $x$-axis

symmetry about the $y$-axis

symmetry about the origin


$$
\begin{aligned}
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& {[r, \pi-\theta] \in \text { graph }} \\
& \Rightarrow \text { symmetric about the } y \text {-axis. }
\end{aligned}
$$

## Symmetry


symmetry about the $x$-axis

symmetry about the $y$-axis

symmetry about the origin


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& \cos [2(\pi-\theta)]=\cos (2 \pi-2 \theta)=\cos 2 \theta \\
& \Rightarrow \text { if }[r, \theta] \in \text { graph, then } \\
& {[r, \pi-\theta] \in \text { graph }} \\
& \Rightarrow \text { symmetric about the } y \text {-axis. }
\end{aligned}
$$

## Symmetry


symmetry about the $x$-axis

symmetry about the $y$-axis

symmetry about the origin


$$
\begin{aligned}
& \text { Lemniscate (ribbon) } r^{2}=\cos 2 \theta \\
& \cos [2(\pi+\theta)]=\cos (2 \pi+2 \theta)=\cos 2 \theta
\end{aligned}
$$

$$
\Rightarrow \text { if }[r, \theta] \in \text { graph, then }
$$

$$
[r, \pi+\theta] \in \text { graph }
$$

$$
\Rightarrow \text { symmetric about the origin. }
$$

## Symmetry


symmetry about the $x$-axis

symmetry about the $y$-axis

symmetry about the origin


$$
\begin{aligned}
& \text { Lemniscate (ribbon) } r^{2}=\cos 2 \theta \\
& \cos [2(\pi+\theta)]=\cos (2 \pi+2 \theta)=\cos 2 \theta \\
& \Rightarrow \text { if }[r, \theta] \in \text { graph, then } \\
& {[r, \pi+\theta] \in \text { graph }} \\
& \Rightarrow \text { symmetric about the origin. }
\end{aligned}
$$

## Symmetry


symmetry about the $x$-axis

symmetry about the $y$-axis

symmetry about the origin


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& \text { Lemniscate (ribbon) } r^{2}=\cos 2 \theta \\
& \cos [2(\pi+\theta)]=\cos (2 \pi+2 \theta)=\cos 2 \theta \\
& \Rightarrow \text { if }[r, \theta] \in \text { graph, then } \\
& {[r, \pi+\theta] \in \text { graph }} \\
& \Rightarrow \text { symmetric about the origin. }
\end{aligned}
$$



$$
r^{2}=4 \sin 2 \theta
$$


$r^{2}=4 \cos 2 \theta$

Lemniscate $r^{2}=a \sin 2 \theta$
$\sin [2(\pi+\theta)]=\sin (2 \pi+2 \theta)=\sin 2 \theta$
$\Rightarrow$ if $[r, \theta] \in$ graph, then $[r, \pi+\theta] \in$ graph

## Lemniscates (Ribbons) $r^{2}=a \sin 2 \theta, r^{2}=a \cos 2 \theta$



Lemniscate $r^{2}=a \sin 2 \theta$
$\sin [2(\pi+\theta)]=\sin (2 \pi+2 \theta)=\sin 2 \theta$
$\Rightarrow$ if $[r, \theta] \in$ graph, then $[r, \pi+\theta] \in$ graph

## Lemniscates (Ribbons) $r^{2}=a \sin 2 \theta, r^{2}=a \cos 2 \theta$



$r^{2}=4 \cos 2 \theta$

Lemniscate $r^{2}=a \sin 2 \theta$
$\sin [2(\pi+\theta)]=\sin (2 \pi+2 \theta)=\sin 2 \theta$
$\Rightarrow$ if $[r, \theta] \in$ graph, then $[r, \pi+\theta] \in$ graph

## Lemniscates (Ribbons) $r^{2}=a \sin 2 \theta, r^{2}=a \cos 2 \theta$



$r^{2}=4 \cos 2 \theta$
Lemniscate $r^{2}=a \sin 2 \theta$
$\sin [2(\pi+\theta)]=\sin (2 \pi+2 \theta)=\sin 2 \theta$
$\Rightarrow$ if $[r, \theta] \in$ graph, then $[r, \pi+\theta] \in$ graph
$\Rightarrow$ symmetric about the origin.

## Quiz

## Quiz

1. $r=\sec \theta$ is sym. about
(a) $x$-axis,
(b) $y$-axis,
(c) origin.
2. $r=2 \sin \theta$ is a
(a) line,
(b) circle, (c) lemniscate

## Spiral of Archimedes $r=\theta, \theta \geq 0$

$$
\left[\frac{3}{4} \pi, \frac{3}{4} \pi\right]
$$

The curve is a nonending spiral. Here it is shown in detail from $\theta=0$ to $\theta=2 \pi$.

## Limaçon (Snail): $r=1-2 \cos \theta$



| $\theta$ | 0 | $\pi / 4$ | $\pi / 3$ | $\pi / 2$ | $2 \pi / 3$ | $3 \pi / 4$ | $\pi$ | $5 \pi / 4$ | $4 \pi / 3$ | $3 \pi / 2$ | $5 \pi / 3$ | $7 \pi / 4$ | $2 \pi$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $r$ | -1 | -0.41 | 0 | 1 | 2 | 2.41 | 3 | 2.41 | 2 | 1 | 0 | -0.41 | -1 |

- $r=0$ at $\theta=\frac{1}{3} \pi, \frac{5}{3} \pi ; \quad|r|$ is a local maximum at $\theta=0, \pi, 2 \pi$. - Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.


## Limaçon (Snail): $r=1-2 \cos \theta$

## $[-1,0]$


$0 \leq \theta \leq \frac{1}{3} \pi$


$0 \leq \theta \leq \pi$


| $\theta$ | 0 | $\pi / 4$ | $\pi / 3$ | $\pi / 2$ | $2 \pi / 3$ | $3 \pi / 4$ | $\pi$ | $5 \pi / 4$ | $4 \pi / 3$ | $3 \pi / 2$ | $5 \pi / 3$ | $7 \pi / 4$ | $2 \pi$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $r$ | -1 | -0.41 | 0 | 1 | 2 | 2.41 | 3 | 2.41 | 2 | 1 | 0 | -0.41 | -1 |

- $r=0$ at $\theta=\frac{1}{3} \pi, \frac{5}{3} \pi ; \quad|r|$ is a local maximum at $\theta=0, \pi, 2 \pi$.
- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$ symmetric about the $x$-axis.


## Limaçon (Snail): $r=1-2 \cos \theta$


polar axis
$0 \leq \theta \leq \frac{1}{3} \pi$


$0 \leq \theta \leq \pi$


| $\theta$ | 0 | $\pi / 4$ | $\pi / 3$ | $\pi / 2$ | $2 \pi / 3$ | $3 \pi / 4$ | $\pi$ | $5 \pi / 4$ | $4 \pi / 3$ | $3 \pi / 2$ | $5 \pi / 3$ | $7 \pi / 4$ | $2 \pi$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $r$ | -1 | -0.41 | 0 | 1 | 2 | 2.41 | 3 | 2.41 | 2 | 1 | 0 | -0.41 | -1 |

- $r=0$ at $\theta=\frac{1}{3} \pi, \frac{5}{3} \pi ; \quad|r|$ is a local maximum at $\theta=0, \pi, 2 \pi$.
- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
o $\cos (-\theta)=\cos \theta \Rightarrow$ if $[r, \theta] \in$ graph, then $[r,-\theta] \in$ graph $\Rightarrow$ symmetric about the $x$-axis.


## Limaçon (Snail): $r=1-2 \cos \theta$


polar axis
$0 \leq \theta \leq \frac{1}{3} \pi$


$0 \leq \theta \leq \pi$


| $\theta$ | 0 | $\pi / 4$ | $\pi / 3$ | $\pi / 2$ | $2 \pi / 3$ | $3 \pi / 4$ | $\pi$ | $5 \pi / 4$ | $4 \pi / 3$ | $3 \pi / 2$ | $5 \pi / 3$ | $7 \pi / 4$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $2 \pi$ |  |  |  |  |  |  |  |  |  |  |  |  |
| $r$ | -1 | -0.41 | 0 | 1 | 2 | 2.41 | 3 | 2.41 | 2 | 1 | 0 | -0.41 |

- $r=0$ at $\theta=\frac{1}{3} \pi, \frac{5}{3} \pi ; \quad|r|$ is a local maximum at $\theta=0, \pi, 2 \pi$.
- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
- $\cos (-\theta)=\cos \theta \Rightarrow$ if $[r, \theta] \in$ graph, then $[r,-\theta] \in$ graph $\Rightarrow$ symmetric about the $x$-axis.


## Limaçons (Snails): $r=a+b \cos \theta$


$r=3+\cos \theta$ convex limaçon

$r=\frac{3}{2}+\cos \theta$ limaçon
with a dimple

$r=1+\cos \theta$ cardioid

$r=\frac{1}{2}+\cos \theta$ limaçon with an inner loop

The general shape of the curve depends on the relative magnitudes of $|a|$ and $|b|$.

## Cardioids (Heart-Shaped): $r=1 \pm \cos \theta, r=1 \pm \sin \theta$


$r=1+\cos \theta$

$r=1+\sin \theta$

$r=1-\cos \theta$

$r=1-\sin \theta$

Each change

$$
\cos \theta \rightarrow \sin \theta \rightarrow-\cos \theta \rightarrow-\sin \theta
$$

represents a counterclockwise rotation by $\frac{1}{2} \pi$ radians.

- Rotation by $\frac{1}{2} \pi: r=1+\cos \left(\theta-\frac{1}{2} \pi\right)=1+\sin \theta$.



## Cardioids (Heart-Shaped): $r=1 \pm \cos \theta, r=1 \pm \sin \theta$


$r=1+\cos \theta$

$r=1+\sin \theta$

$r=1-\cos \theta$

$r=1-\sin \theta$

Each change

$$
\cos \theta \rightarrow \sin \theta \rightarrow-\cos \theta \rightarrow-\sin \theta
$$

represents a counterclockwise rotation by $\frac{1}{2} \pi$ radians.

- Rotation by $\frac{1}{2} \pi: r=1+\cos \left(\theta-\frac{1}{2} \pi\right)=1+\sin \theta$.
- Rotation by $\frac{1}{2} \pi: r=1+\sin \left(\theta-\frac{1}{2} \pi\right)=1-\cos \theta$.



## Cardioids (Heart-Shaped): $r=1 \pm \cos \theta, r=1 \pm \sin \theta$


$r=1+\cos \theta$

$r=1+\sin \theta$

$r=1-\cos \theta$

$r=1-\sin \theta$

Each change

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\cos \theta \rightarrow \sin \theta \rightarrow-\cos \theta \rightarrow-\sin \theta
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## Quiz

## Quiz

3. $r=\sin \theta$ is the rotation by $\frac{\pi}{2}$ of:

$$
\text { (a) } r=\cos \theta, \quad \text { (b) } r=-\sin \theta, \quad \text { (c) } r=-\cos \theta \text {. }
$$

4. Today is
(a) Feb. 19,
(b) Feb. 20,
(c) Feb. 21.


- $r=0$ at $\theta=\frac{\pi}{4}, \frac{3 \pi}{4}, \frac{5 \pi}{4}, \frac{7 \pi}{4}$;
$|r|$ is a local maximum at $\theta=0, \frac{\pi}{2}, \pi, \frac{3 \pi}{2}, 2 \pi$.
- Sketch the curve in 8 stages.
- $\cos [2(-\theta)]=\cos 2 \theta, \cos [2(\pi \pm \theta)]=\cos 2 \theta$
$\qquad$



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Petal Curve: $r=\cos 2 \theta$




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## Petal Curves: $r=a \cos n \theta, r=a \sin n \theta$




- If $n$ is odd, there are $n$ petals.
- If $n$ is even, there are $2 n$ petals.


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## Intersections: $r=a(1-\cos \theta)$ and $r=a(1+\cos \theta)$


$r=a(1-\cos \theta)$ and $r=a(1+\cos \theta) \Rightarrow r=a$ and $\cos \theta=0$ $\Rightarrow r=a$ and $\theta=\frac{\pi}{2}+n \pi \Rightarrow\left[a, \frac{\pi}{2}+n \pi\right] \in$ intersection $\Rightarrow n$ even, $\left[a, \frac{\pi}{2}+n \pi\right]=\left[a, \frac{\pi}{2}\right] ; n$ odd, $\left[a, \frac{\pi}{2}+n \pi\right]=\left[a, \frac{3 \pi}{2}\right]$ - Two intersection points: $\left[a, \frac{\pi}{2}\right]=(0, a)$ and $\left[a, \frac{3 \pi}{2}\right]=(0,-a)$

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## Outline

- Polar Coordinates
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- Relation to Rectangular Coordinates
- Symmetry
- Graphing in Polar Coordinates
- Spiral
- Limaçons
- Flowers
- Intersections

