Lecture 13

Section 9.5 Area in Polar Coordinates

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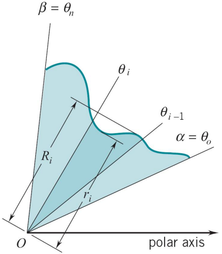
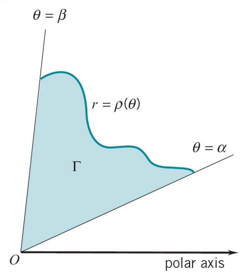
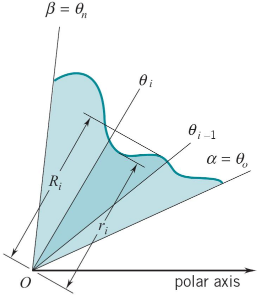


Area of a Polar Region





The area of the polar region Γ generated by

$$r = \rho(\theta), \quad \alpha \le \theta \le \beta$$

is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} [\rho(\theta)]^2 \, d\theta$$

Proof

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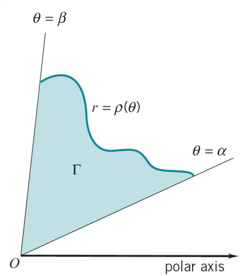

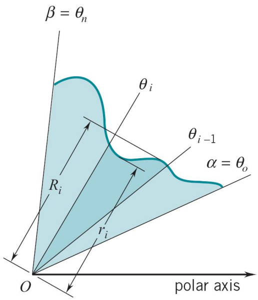

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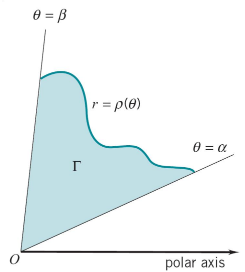

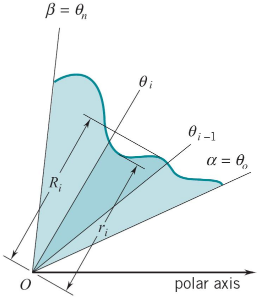

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Let $P=\{\theta_0,\theta_1,\cdots,\theta_n\}$ be a partition of $[\alpha,\beta]$. Set $r_i=\min\limits_{\alpha\le\theta\le\beta}\rho(\theta)$ and $R_i=\max\limits_{\alpha\le\theta\le\beta}\rho(\theta)$. Then

$$\frac{1}{2}r_i^2\Delta\theta_i\le A_i\le\frac{1}{2}R_i^2\Delta\theta_i$$

Summing from $i=1$ to $i=n$ yields

$$L_f(P)\le A\le U_f(P)\quad\text{with}\quad f(\theta)=\frac{1}{2}\left[\rho(\theta)\right]^2$$

Since P is arbitrary, we conclude

$$A=\int_\alpha^\beta f(\theta)\,d\theta=\int_\alpha^\beta\frac{1}{2}\left[\rho(\theta)\right]^2 d\theta$$

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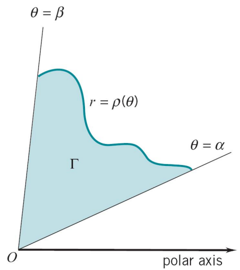

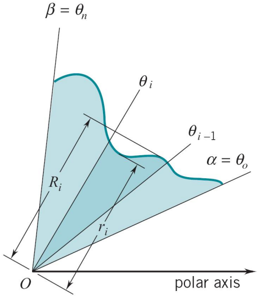

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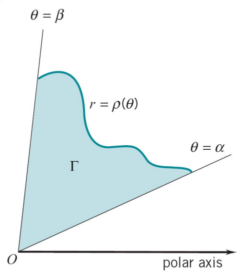

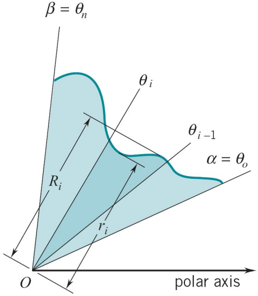

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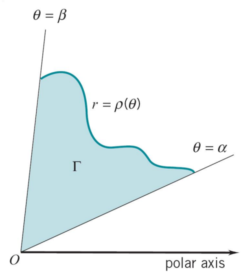

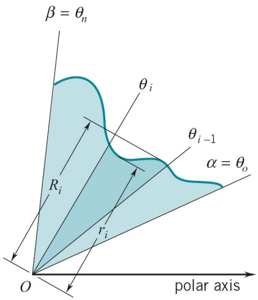

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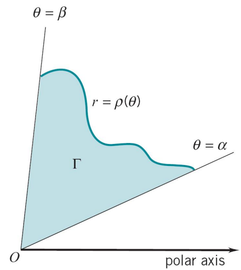

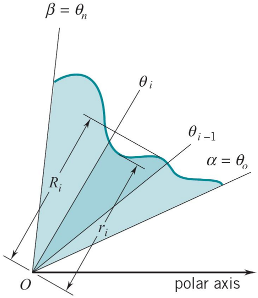

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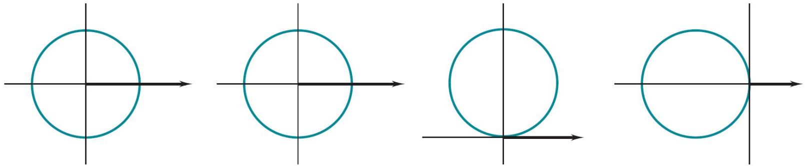

$r = 2$ $r = -2$ $r = 4 \sin\theta$ $r = -4 \cos\theta$

Circle in Polar Coordinates

$$r = a, \quad 0 \le \theta \le 2\pi$$

$$A = \int_\alpha^\beta \frac{1}{2}\left[\rho(\theta)\right]^2 d\theta = \int_0^{2\pi} \frac{1}{2}\left[a\right]^2 d\theta = \frac{1}{2}a^2 \cdot 2\pi = \pi a^2$$

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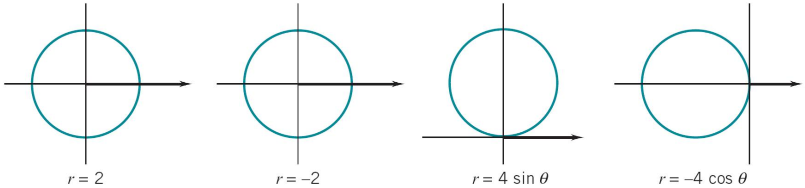

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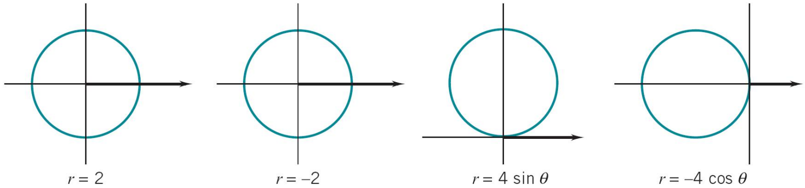

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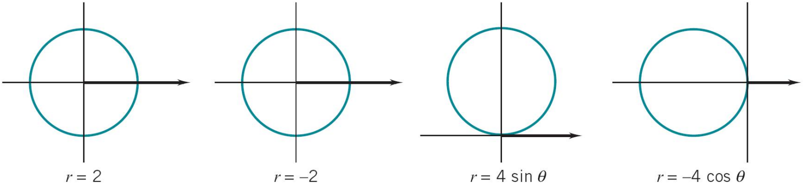

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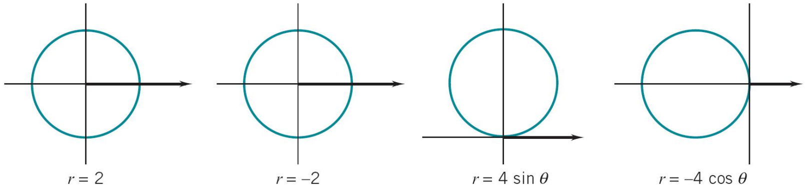

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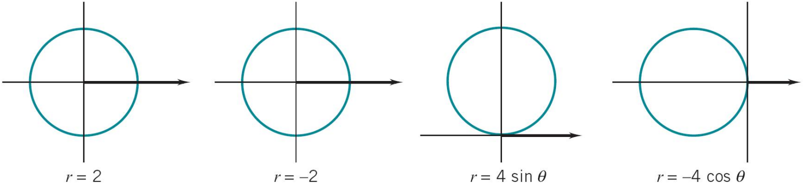

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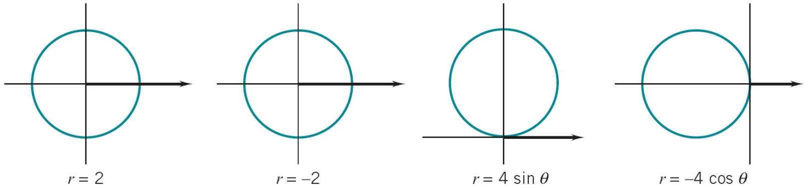

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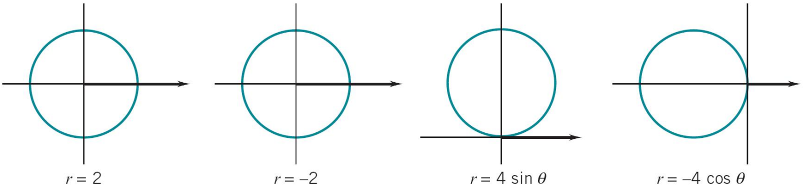

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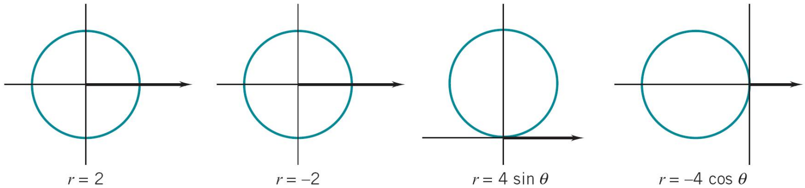

Circle in Polar Coordinates

$$r = 2a\sin\theta, \quad 0 \le \theta \le \pi$$

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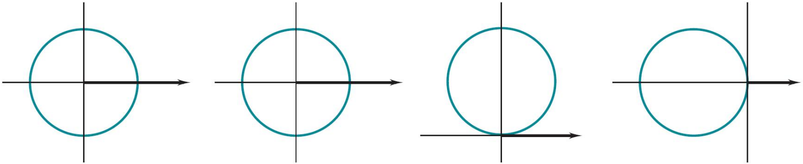

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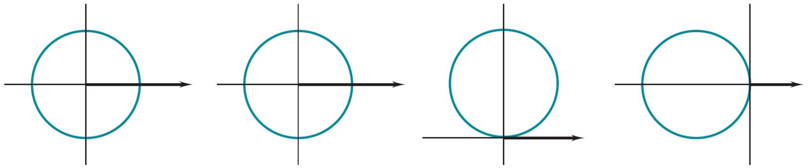

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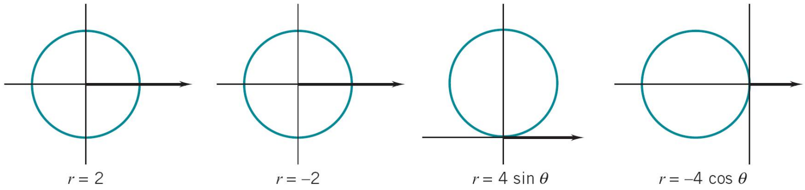

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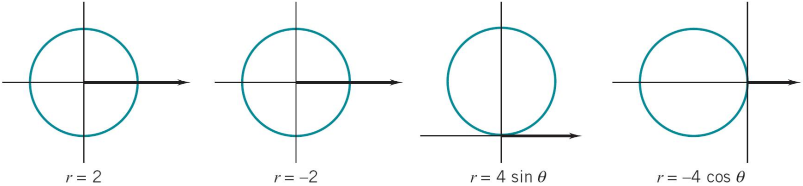

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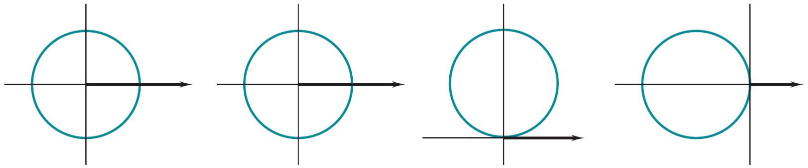

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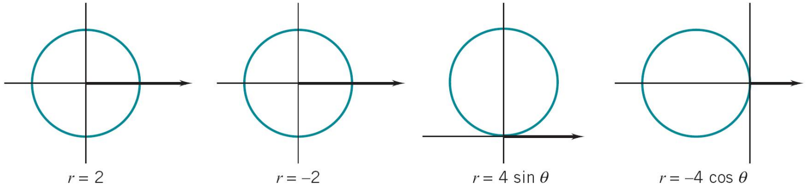

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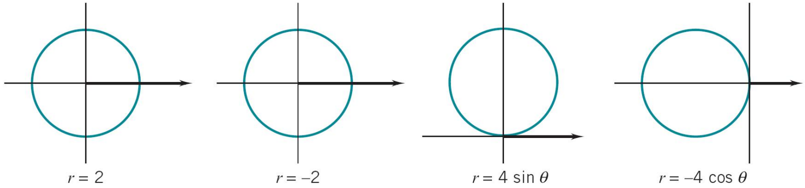

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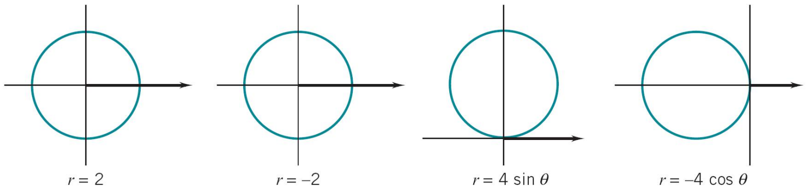

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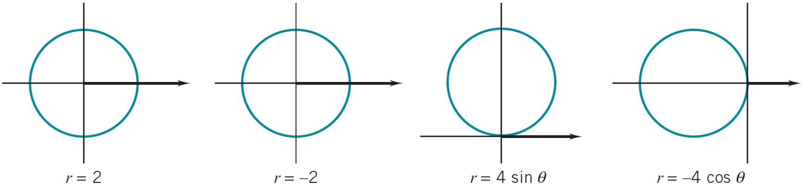
$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left[\rho(\theta)\right]^2 d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}\left[-2a\cos\theta\right]^2 d\theta = 2a^2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2\theta\, d\theta$$

$$= 2a^2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = 2a^2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}\, d\theta = 2a^2 \cdot \frac{\pi}{2} = \pi a^2$$

Area of a Circle of Radius a: $A = \pi a^2$



Circle in Polar Coordinates

$$r = -2a\cos\theta, \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

$$A = \int_\alpha^\beta \frac{1}{2}\left[\rho(\theta)\right]^2 d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}\left[-2a\cos\theta\right]^2 d\theta = 2a^2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2\theta\, d\theta$$

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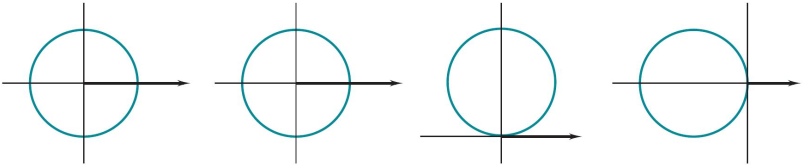

Circle in Polar Coordinates

$$r = -2a\cos\theta, \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

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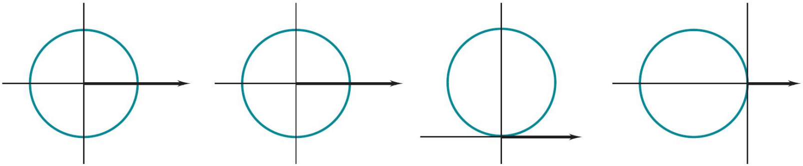

Circle in Polar Coordinates

$$r = -2a\cos\theta, \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

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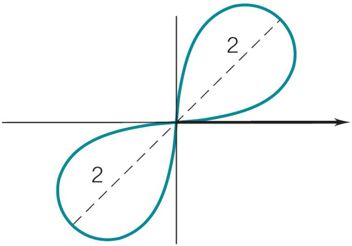
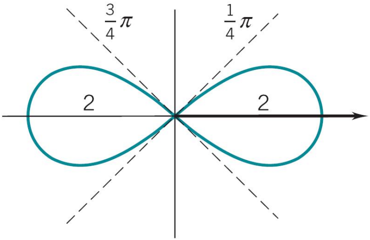
Area of a Lemniscate (Ribbon): $A = a^2$



$r^2 = 4 \sin 2\theta$



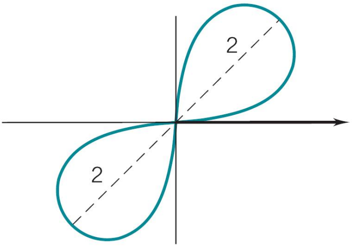
$r^2 = 4 \cos 2\theta$

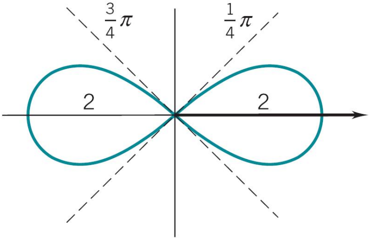
Ribbon

Sketch $r^2 = a^2 \cos 2\theta$ in 4 stages: $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{4}]$, $[\frac{3\pi}{4}, \pi]$

$$A = 4\int_\alpha^\beta \frac{1}{2}[\rho(\theta)]^2 d\theta = 4\int_0^{\frac{\pi}{4}} \frac{1}{2}(a^2 \cos 2\theta)\, d\theta = 2a^2\left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = a^2$$

Area of a Lemniscate (Ribbon): $A = a^2$





Ribbon

Sketch $r^2 = a^2 \cos 2\theta$ in 4 stages: $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{4}]$, $[\frac{3\pi}{4}, \pi]$

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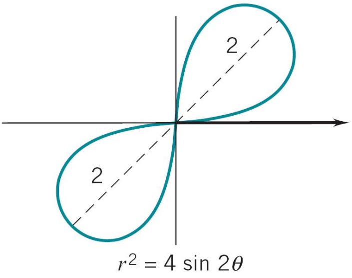

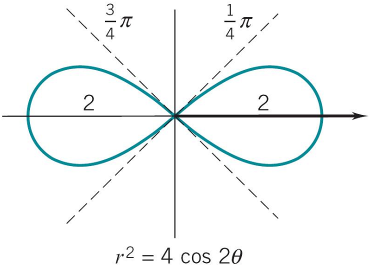

Ribbon

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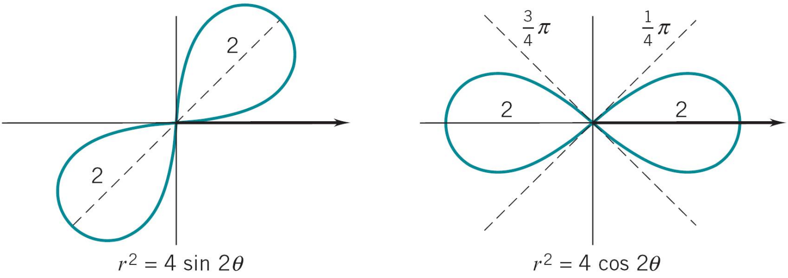

Ribbon

Sketch $r^2 = a^2 \cos 2\theta$ in 4 stages: $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{4}]$, $[\frac{3\pi}{4}, \pi]$

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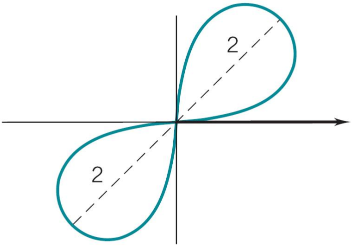
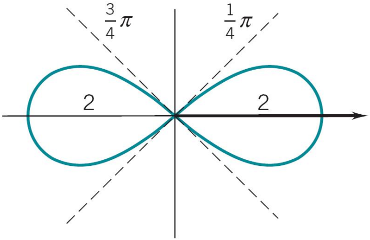
Area of a Lemniscate (Ribbon): $A = a^2$



$r^2 = 4 \sin 2\theta$



$r^2 = 4 \cos 2\theta$

Ribbon

Sketch $r^2 = a^2 \sin 2\theta$ in 4 stages: $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{4}]$, $[\frac{3\pi}{4}, \pi]$

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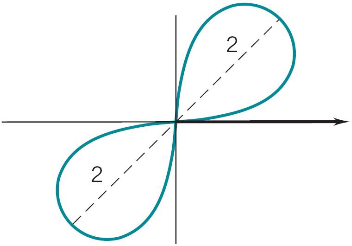
Area of a Lemniscate (Ribbon): $A = a^2$



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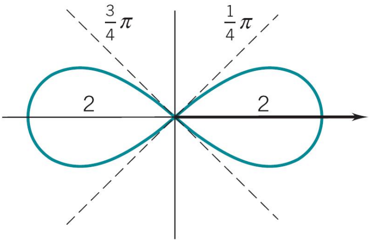

$r^2 = 4 \cos 2\theta$

Ribbon

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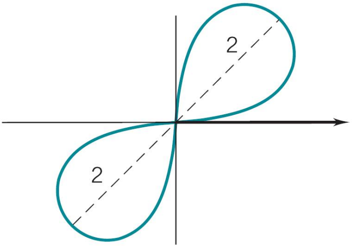
Area of a Lemniscate (Ribbon): $A = a^2$



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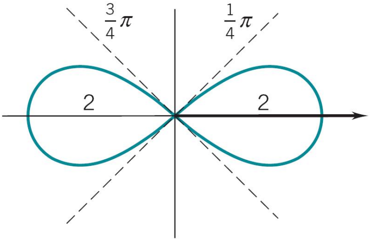

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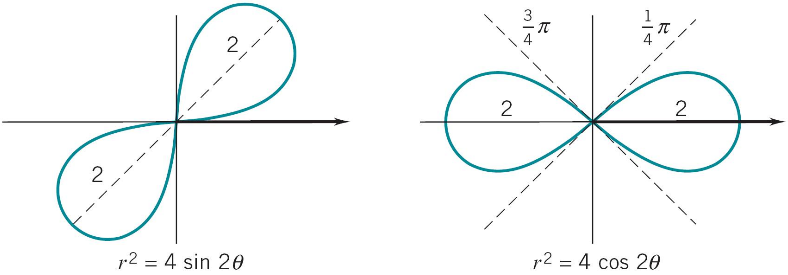
Area of a Lemniscate (Ribbon): $A = a^2$



Ribbon

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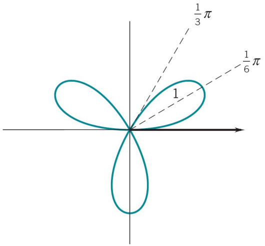
Quiz

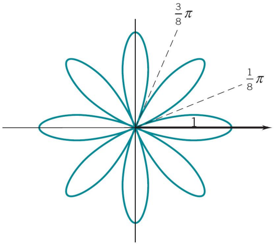
Quiz

1. $r = 2a\sin\theta$ is a (a) line, (b) circle, (c) lemniscate.
2. $r^2 = a^2\sin 2\theta$ is a (a) line, (b) circle, (c) lemniscate.

Area of a Flower: $A = \frac{\pi}{4}, \frac{\pi}{2}$



$r = \sin 3\theta$



$r = \cos 4\theta$

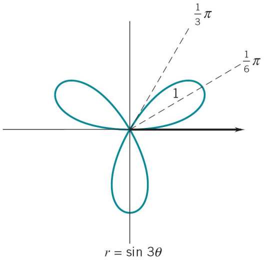
Flower

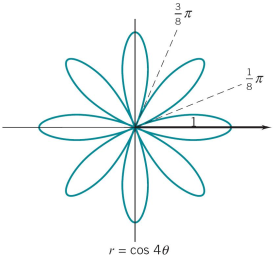
Sketch $r = \sin 3\theta$ in 6 stages: $[0, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{3}]$, $\cdots$, $[\frac{2\pi}{3}, \frac{5\pi}{6}]$, $[\frac{5\pi}{6}, \pi]$

$$A = 6\int_{\alpha}^{\beta} \frac{1}{2}\left[\rho(\theta)\right]^2 d\theta = 6\int_{0}^{\frac{\pi}{6}} \frac{1}{2}\left[\sin 3\theta\right]^2 d\theta$$

$$= 3\int_{0}^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2}\cos 6\theta\right) d\theta = \frac{3}{2}\left[\theta - \frac{1}{6}\sin 6\theta\right]_{0}^{\frac{\pi}{6}} = \frac{\pi}{4}$$

Area of a Flower: $A = \frac{\pi}{4}, \frac{\pi}{2}$





Flower

Sketch $r = \sin 3\theta$ in 6 stages: $[0, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{3}]$, $\cdots$, $[\frac{2\pi}{3}, \frac{5\pi}{6}]$, $[\frac{5\pi}{6}, \pi]$

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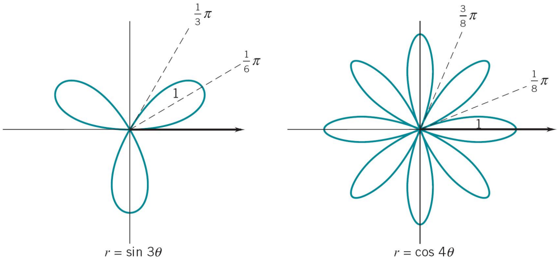
Area of a Flower: $A = \frac{\pi}{4}, \frac{\pi}{2}$



Flower

Sketch $r = \sin 3\theta$ in 6 stages: $[0, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{3}]$, $\cdots$, $[\frac{2\pi}{3}, \frac{5\pi}{6}]$, $[\frac{5\pi}{6}, \pi]$

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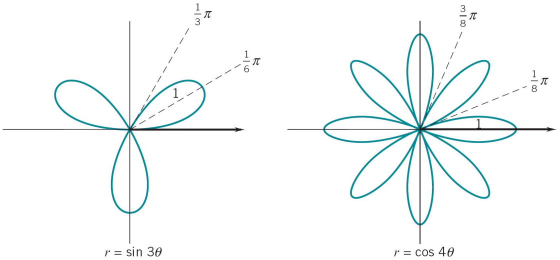
Area of a Flower: $A = \frac{\pi}{4}, \frac{\pi}{2}$



Flower

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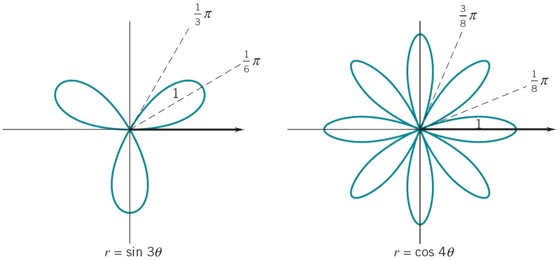
Area of a Flower: $A = \frac{\pi}{4}, \frac{\pi}{2}$



Flower

Sketch $r = \sin 3\theta$ in 6 stages: $[0, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{3}]$, $\cdots$, $[\frac{2\pi}{3}, \frac{5\pi}{6}]$, $[\frac{5\pi}{6}, \pi]$

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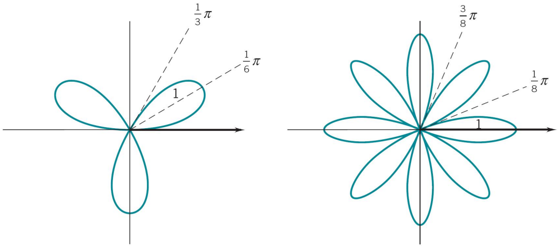
Area of a Flower: $A = \frac{\pi}{4}, \frac{\pi}{2}$



$r = \sin 3\theta$ $r = \cos 4\theta$

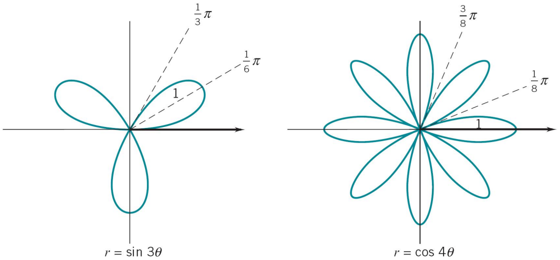
Flower

Sketch $r = \cos 4\theta$ in 16 stages: $[0, \frac{\pi}{8}]$, $[\frac{\pi}{8}, \frac{\pi}{4}]$, $\cdots$, $[\frac{15\pi}{8}, 2\pi]$

$$A = 16\int_{\alpha}^{\beta}\frac{1}{2}\left[\rho(\theta)\right]^2 d\theta = 16\int_{0}^{\frac{\pi}{8}}\frac{1}{2}\left[\cos 4\theta\right]^2 d\theta$$

$$= 8\int_{0}^{\frac{\pi}{8}}\left(\frac{1}{2}+\frac{1}{2}\cos 8\theta\right)d\theta = 4\left[\theta + \frac{1}{8}\sin 8\theta\right]_{0}^{\frac{\pi}{8}} = \frac{\pi}{2}$$

Area of a Flower: $A = \frac{\pi}{4}, \frac{\pi}{2}$



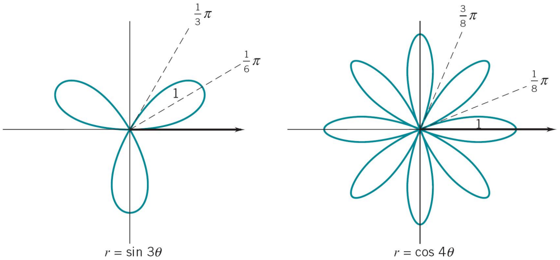
Flower

Sketch $r = \cos 4\theta$ in 16 stages: $[0, \frac{\pi}{8}]$, $[\frac{\pi}{8}, \frac{\pi}{4}]$, $\cdots$, $[\frac{15\pi}{8}, 2\pi]$

$$A = 16\int_{\alpha}^{\beta}\frac{1}{2}\left[\rho(\theta)\right]^2 d\theta = 16\int_0^{\frac{\pi}{8}}\frac{1}{2}\left[\cos 4\theta\right]^2 d\theta$$

$$= 8\int_0^{\frac{\pi}{8}}\left(\frac{1}{2}+\frac{1}{2}\cos 8\theta\right)d\theta = 4\left[\theta + \frac{1}{8}\sin 8\theta\right]_0^{\frac{\pi}{8}} = \frac{\pi}{2}$$

Area of a Flower: $A = \frac{\pi}{4}, \frac{\pi}{2}$



Flower

Sketch $r = \cos 4\theta$ in 16 stages: $[0, \frac{\pi}{8}]$, $[\frac{\pi}{8}, \frac{\pi}{4}]$, $\cdots$, $[\frac{15\pi}{8}, 2\pi]$

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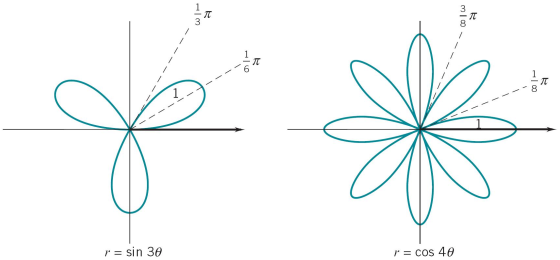
Area of a Flower: $A = \frac{\pi}{4}, \frac{\pi}{2}$



Flower

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$$A = 16\int_{\alpha}^{\beta}\frac{1}{2}\left[\rho(\theta)\right]^2 d\theta = 16\int_0^{\frac{\pi}{8}}\frac{1}{2}\left[\cos 4\theta\right]^2 d\theta$$

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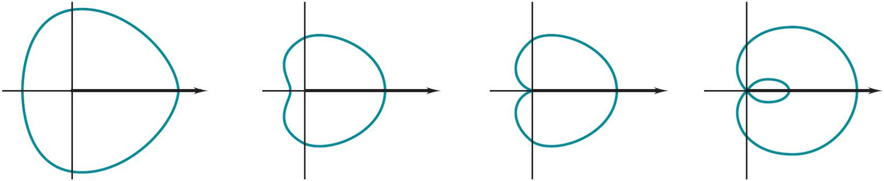
Limaçons (Snails): $r = a + \cos\theta,\ a \geq 1$



$r = 3 + \cos\theta$ convex limaçon

$r = \frac{3}{2} + \cos\theta$ limaçon with a dimple

$r = 1 + \cos\theta$ cardioid

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Flower

Sketch $r = a + \cos\theta$, $a \geq 1$, in 2 stages: $[0, \pi]$, $[\pi, 2\pi]$

$$A = 2\int_0^\pi \frac{1}{2}\left[a + \cos\theta\right]^2 d\theta = \int_0^\pi \left(a^2 + 2a\cos\theta + \cos^2\theta\right) d\theta$$

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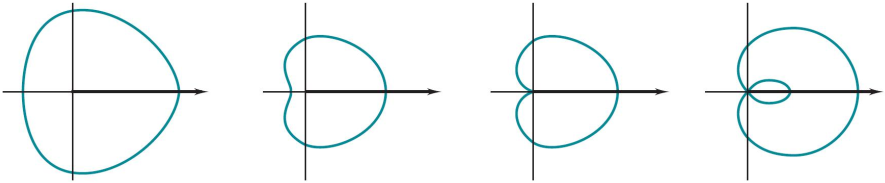
Limaçons (Snails): $r = a + \cos\theta,\ a \geq 1$



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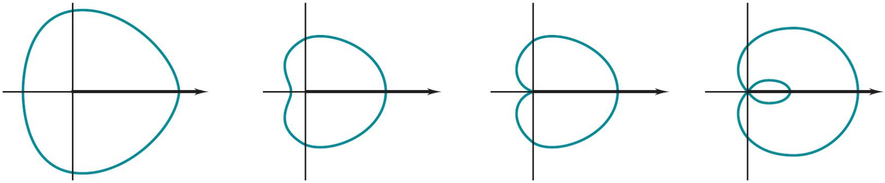

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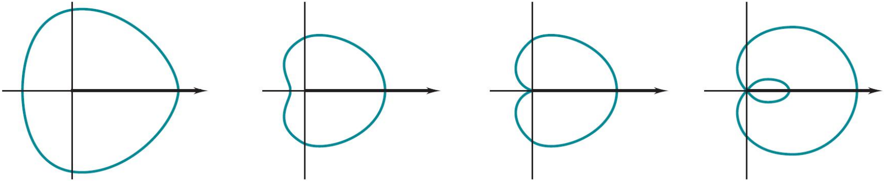

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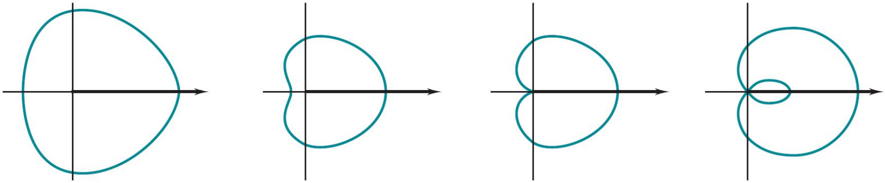

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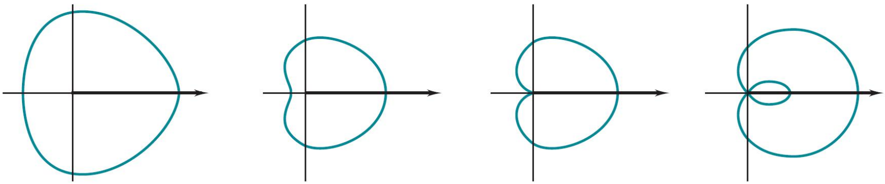

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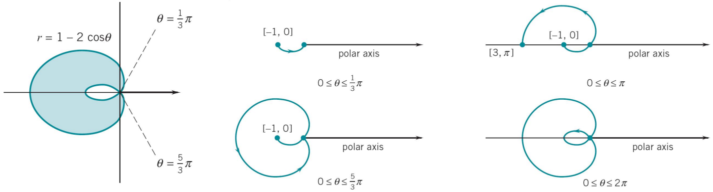

Area between the Inner and Outer Loops

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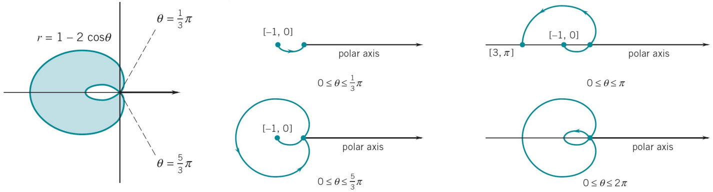

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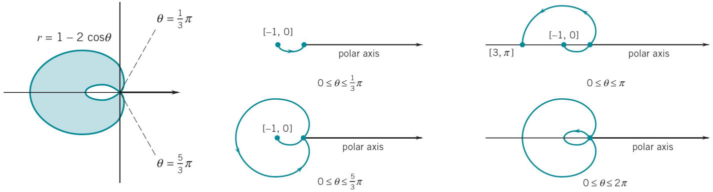

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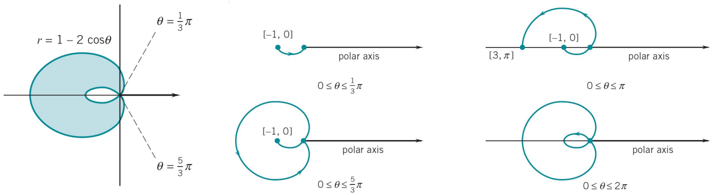

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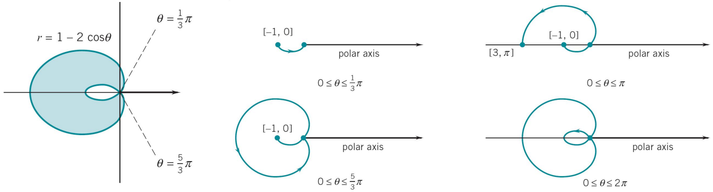

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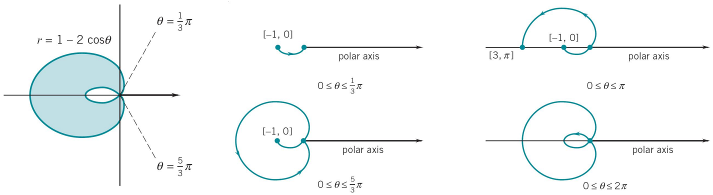

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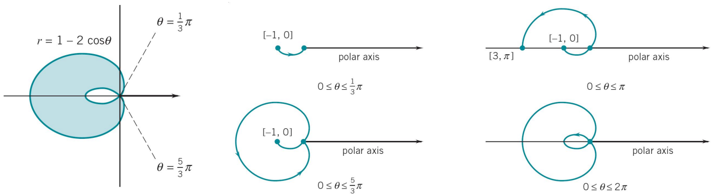

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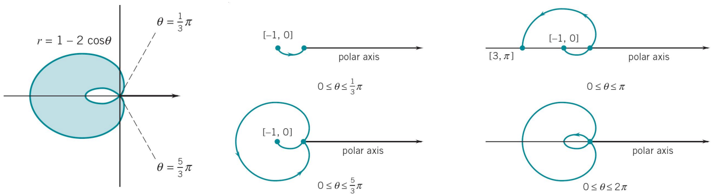

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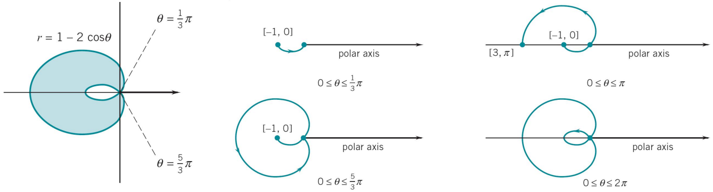

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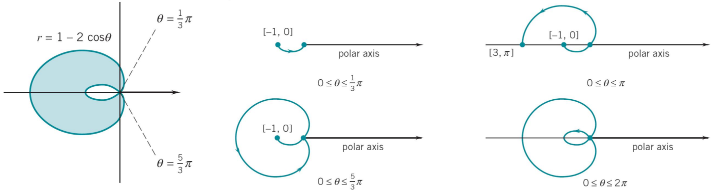

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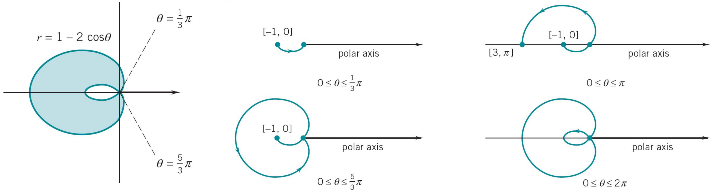

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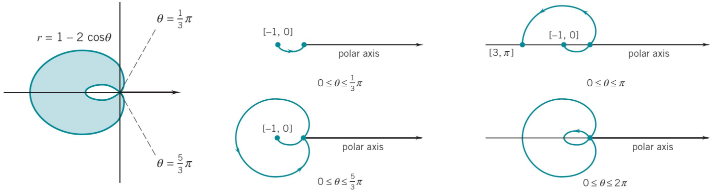

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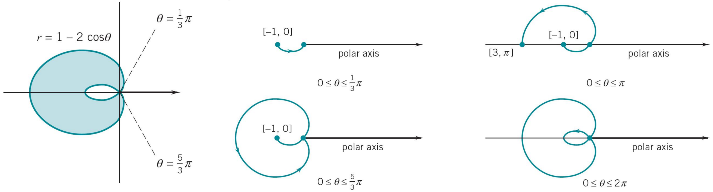

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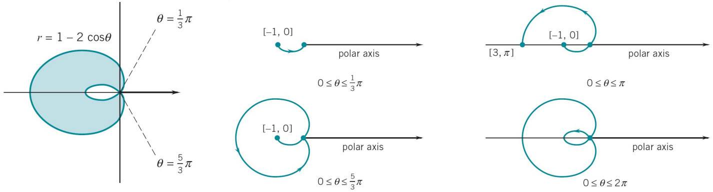

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Area Within Inner Loop: A_{inner}

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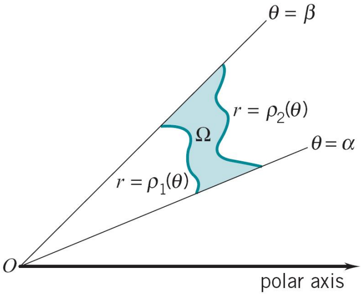
Area between Polar Curves



Area between $r = \rho_1(\theta)$ and $r = \rho_2(\theta)$

$$\text{area of } \Omega = \int_{\alpha_2}^{\beta_2} \frac{1}{2}\left[\rho_2(\theta)\right]^2 d\theta - \int_{\alpha_1}^{\beta_1} \frac{1}{2}\left[\rho_1(\theta)\right]^2 d\theta$$

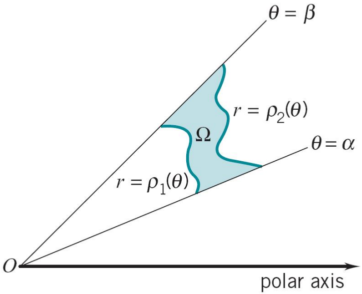
$$= \int_{\alpha}^{\beta} \frac{1}{2}\left(\left[\rho_2(\theta)\right]^2 - \left[\rho_1(\theta)\right]^2\right) d\theta$$

if $\alpha_1 = \alpha_2 = \alpha, \quad \beta_1 = \beta_2 = \beta.$

Remark

Extra care is needed to determine the intervals of θ values (e.g, $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$) over which the outer and inner boundaries of the region are traced out.

Area between Polar Curves



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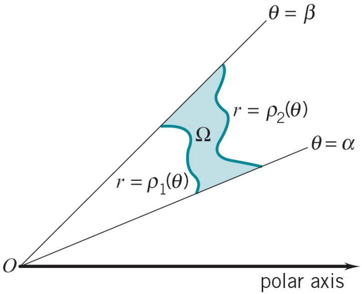

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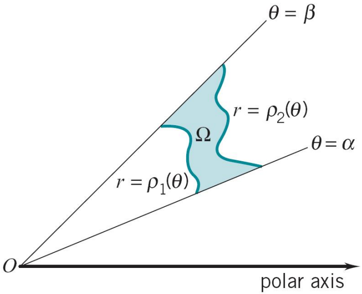

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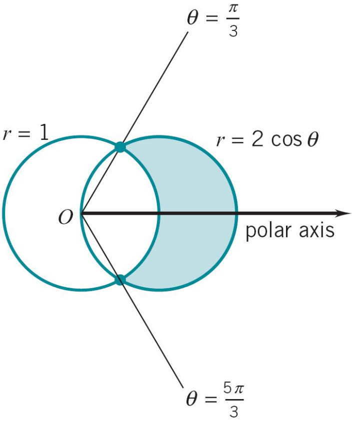
Extra care is needed to determine the intervals of θ values (e.g, $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$) over which the outer and inner boundaries of the region are traced out.

Area between Circles: $r = 2\cos\theta$ and $r = 1$



Area between $r = 2\cos\theta$ and $r = 1$

- The two intersection points: $2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \frac{\pi}{3}, \frac{5\pi}{3} \Rightarrow [\alpha, \beta] = [-\frac{\pi}{3}, \frac{\pi}{3}]$
- By symmetry,

  area of Ω

  $$= \int_\alpha^\beta \frac{1}{2}\left([\rho_2(\theta)]^2 - [\rho_1(\theta)]^2\right) d\theta$$

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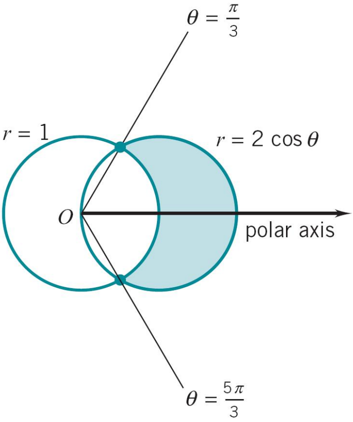

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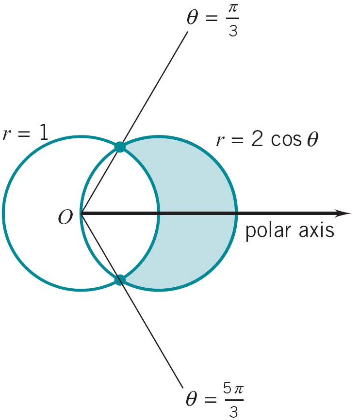

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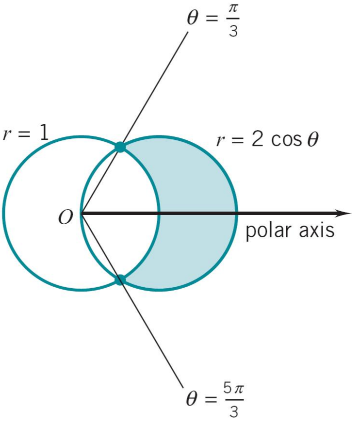
$$= \int_0^{\frac{\pi}{3}} (1 + 2\cos 2\theta)\, d\theta$$

$$= \left[\theta + \sin 2\theta\right]_0^{\frac{\pi}{3}} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

Area between Circles: $r = 2\cos\theta$ and $r = 1$



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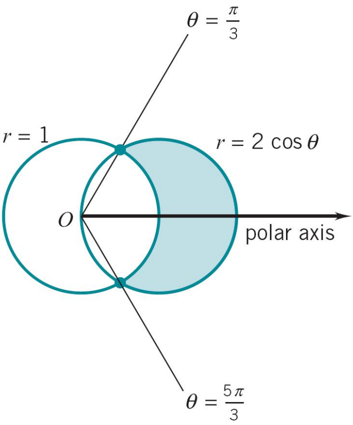

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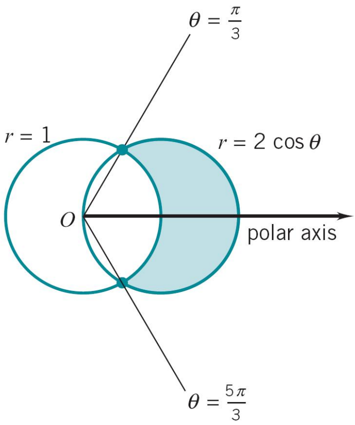

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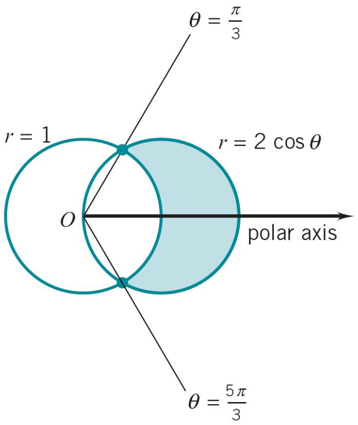

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Quiz

Quiz

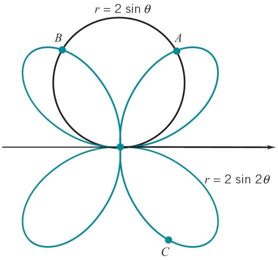
3. area of $r = 2a\sin\theta$ is :

$$(a)\ \pi a^2, \quad (b)\ \frac{1}{2}\pi a^2, \quad (c)\ a^2.$$

4. area of $r^2 = a^2 \sin 2\theta$ is :

$$(a)\ \pi a^2, \quad (b)\ \frac{1}{2}\pi a^2, \quad (c)\ a^2.$$

Area outside Circle $r = 2\sin\theta$ and inside flower $r = 2\sin 2\theta$



- The three intersection points: $2\sin 2\theta = 2\sin\theta \Rightarrow 2\sin\theta\cos\theta = \sin\theta \Rightarrow \sin\theta(2\cos\theta - 1) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \Rightarrow 0$, or $\frac{\pi}{3}$, $\frac{5\pi}{3} \Rightarrow [\alpha, \beta] = [0, \frac{\pi}{3}]$ and $[\alpha, \beta] = [\frac{5\pi}{3}, \pi]$
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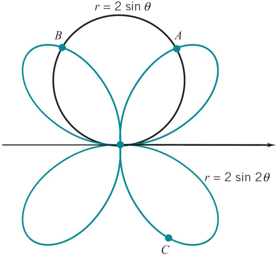
$$A_1 = \int_\alpha^\beta \frac{1}{2}\left([\rho_2(\theta)]^2 - [\rho_1(\theta)]^2\right) d\theta$$

Area outside Circle $r = 2\sin\theta$ and inside flower $r = 2\sin 2\theta$



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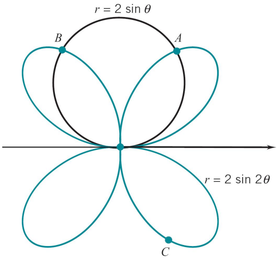

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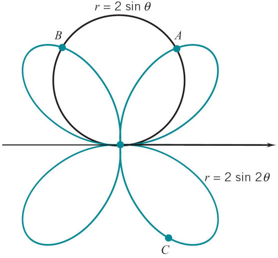
Area outside Circle $r = 2\sin\theta$ and inside flower $r = 2\sin 2\theta$



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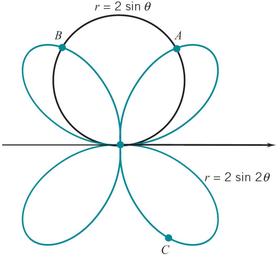

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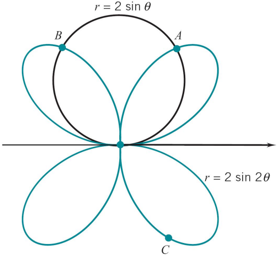

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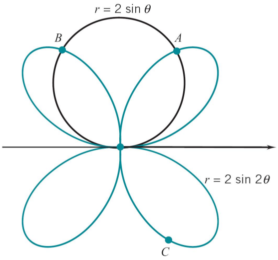
Area outside Circle $r = 2\sin\theta$ and inside flower $r = 2\sin 2\theta$



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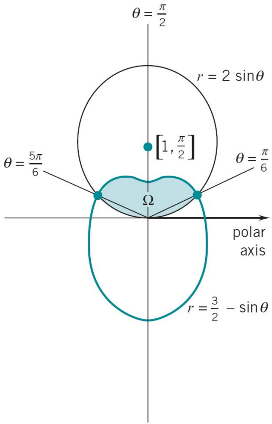
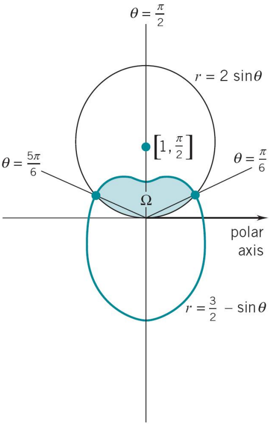
Area between Circle $r = 2\sin\theta$ and Limaçon $r = \frac{3}{2} - \sin\theta$



- The two intersection points:
  $2\sin\theta = \frac{3}{2} - \sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \frac{\pi}{6}, \frac{5\pi}{6}$
- The area can be represented as follows:

$$A = \int_0^{\frac{\pi}{6}} \frac{1}{2}[2\sin\theta]^2\, d\theta$$

Area between Circle $r = 2\sin\theta$ and Limaçon $r = \frac{3}{2} - \sin\theta$



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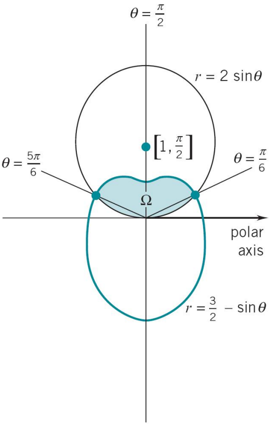

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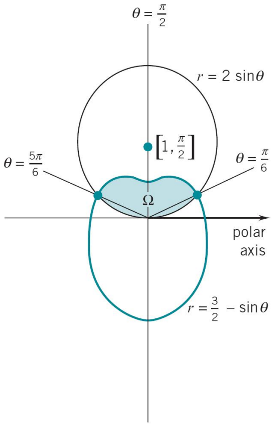
$$A = \int_0^{\frac{\pi}{6}} \frac{1}{2}\left[2\sin\theta\right]^2 d\theta + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\left[\frac{3}{2} - \sin\theta\right]^2 d\theta + \int_{\frac{5\pi}{6}}^{\pi} \frac{1}{2}\left[2\sin\theta\right]^2 d\theta = \cdots = \frac{5}{4}\pi - \frac{15}{8}\sqrt{3}$$

Area between Circle $r = 2\sin\theta$ and Limaçon $r = \frac{3}{2} - \sin\theta$



- The two intersection points:
  $2\sin\theta = \frac{3}{2} - \sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \frac{\pi}{6}, \frac{5\pi}{6}$
- The area can be represented as follows:

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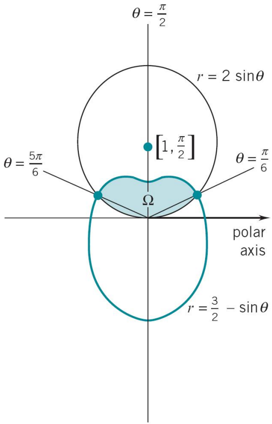

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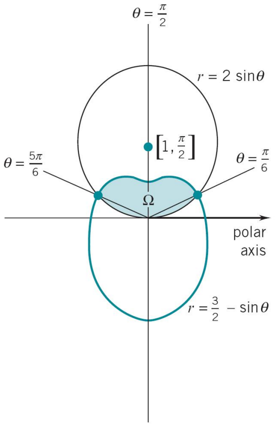

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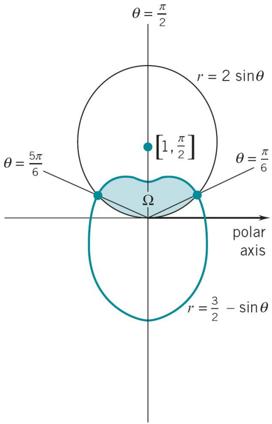

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Outline

- Area of a Polar Region
  - Basic Polar Area
  - Circles
  - Ribbons
  - Flowers
  - Limaçons
- Area between Polar Curves
  - Between Polar Curves
  - Between Circles
  - Between Circle and Flower
  - Between Circle and Limaçon