## Lecture 13 <br> Section 9.5 Area in Polar Coordinates

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## Area of a Polar Region




## Area of a Polar Region



The area of the polar region 「 generated by

$$
r=\rho(\theta), \quad \alpha \leq \theta \leq \beta
$$

is

$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta
$$



## Area of a Polar Region


$\beta=\theta_{n}$


The area of the polar region $\Gamma$ generated by

$$
r=\rho(\theta), \quad \alpha \leq \theta \leq \beta
$$

is

$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta
$$

## Proof

Let $P=\left\{\theta_{0}, \theta_{1}, \cdots, \theta_{n}\right\}$ be a partition of $[\alpha, \beta]$. Set $r_{i}=\min _{\alpha<\theta<\beta} \rho(\theta)$ and $R_{i}=\max _{\alpha<\theta<\beta} \rho(\theta)$. Then

## Area of a Polar Region



## Area of a Polar Region



## Area of a Polar Region



## Area of a Polar Region




## Area of a Circle of Radius a: $A=\pi a^{2}$


$r=2$

$r=-2$

$r=4 \sin \theta$


## Circle in Polar Coordinates

$$
r=a, \quad 0 \leq \theta \leq 2 \pi
$$

$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{2 \pi} \frac{1}{2}[a]^{2} d \theta
$$


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circle in Polar Coordinates

$$
r=a, \quad 0 \leq \theta \leq 2 \pi
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r=a, \quad 0 \leq \theta \leq 2 \pi
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$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{2 \pi} \frac{1}{2}[a]^{2} d \theta=\frac{1}{2} a^{2} \cdot 2 \pi=\pi a^{2}
$$


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circle in Polar Coordinates

$$
r=a, \quad 0 \leq \theta \leq 2 \pi
$$

$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{2 \pi} \frac{1}{2}[a]^{2} d \theta=\frac{1}{2} a^{2} \cdot 2 \pi=\pi a^{2}
$$

## Area of a Circle of Radius a: $A=\pi a^{2}$


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circle in Polar Coordinates

$$
r=-a, \quad 0 \leq \theta \leq 2 \pi
$$

$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{2 \pi} \frac{1}{2}[-a]^{2} d \theta
$$


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circle in Polar Coordinates

$$
r=-a, \quad 0 \leq \theta \leq 2 \pi
$$

$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{2 \pi} \frac{1}{2}[-a]^{2} d \theta
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$r=2$

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A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{2 \pi} \frac{1}{2}[-a]^{2} d \theta=\frac{1}{2} a^{2} \cdot 2 \pi=\pi a^{2}
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$r=2$

$r=-2$

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## Circle in Polar Coordinates

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r=-a, \quad 0 \leq \theta \leq 2 \pi
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A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{2 \pi} \frac{1}{2}[-a]^{2} d \theta=\frac{1}{2} a^{2} \cdot 2 \pi=\pi a^{2}
$$


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

Circle in Polar Coordinates

$$
r=2 a \sin \theta, \quad 0 \leq \theta \leq \pi
$$



$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

Circle in Polar Coordinates

$$
r=2 a \sin \theta, \quad 0 \leq \theta \leq \pi
$$

$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{\pi} \frac{1}{2}[2 a \sin \theta]^{2} d \theta
$$


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

Circle in Polar Coordinates

$$
r=2 a \sin \theta, \quad 0 \leq \theta \leq \pi
$$

$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{\pi} \frac{1}{2}[2 a \sin \theta]^{2} d \theta=2 a^{2} \int_{0}^{\pi} \sin ^{2} \theta d \theta
$$


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

Circle in Polar Coordinates

$$
r=2 a \sin \theta, \quad 0 \leq \theta \leq \pi
$$

$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{\pi} \frac{1}{2}[2 a \sin \theta]^{2} d \theta=2 a^{2} \int_{0}^{\pi} \sin ^{2} \theta d \theta
$$

$=2 a^{2} \int_{0}^{\pi}\left(\frac{1}{2}\right.$ $\square$


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

Circle in Polar Coordinates

$$
r=2 a \sin \theta, \quad 0 \leq \theta \leq \pi
$$

$$
\begin{aligned}
A & =\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{\pi} \frac{1}{2}[2 a \sin \theta]^{2} d \theta=2 a^{2} \int_{0}^{\pi} \sin ^{2} \theta d \theta \\
& =2 a^{2} \int_{0}^{\pi}\left(\frac{1}{2}-\frac{1}{2} \cos 2 \theta\right) d \theta=2 a^{2} \int_{0}^{\pi} \frac{1}{2} d \theta=2 a^{2} \cdot \frac{\pi}{2}=\pi
\end{aligned}
$$


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

Circle in Polar Coordinates

$$
r=2 a \sin \theta, \quad 0 \leq \theta \leq \pi
$$

$$
\begin{aligned}
A & =\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{0}^{\pi} \frac{1}{2}[2 a \sin \theta]^{2} d \theta=2 a^{2} \int_{0}^{\pi} \sin ^{2} \theta d \theta \\
& =2 a^{2} \int_{0}^{\pi}\left(\frac{1}{2}-\frac{1}{2} \cos 2 \theta\right) d \theta=2 a^{2} \int_{0}^{\pi} \frac{1}{2} d \theta=2 a^{2} \cdot \frac{\pi}{2}=\pi a^{2}
\end{aligned}
$$

## Area of a Circle of Radius a: $A=\pi a^{2}$


$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circle in Polar Coordinates

$$
r=-2 a \cos \theta, \quad \frac{\pi}{2} \leq \theta \leq \frac{3 \pi}{2}
$$



$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circle in Polar Coordinates

$$
r=-2 a \cos \theta, \quad \frac{\pi}{2} \leq \theta \leq \frac{3 \pi}{2}
$$

$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \frac{1}{2}[-2 a \cos \theta]^{2} d \theta
$$



$r=2$

$r=-2$

$r=4 \sin \theta$

$r=-4 \cos \theta$

## Circle in Polar Coordinates

$$
r=-2 a \cos \theta, \quad \frac{\pi}{2} \leq \theta \leq \frac{3 \pi}{2}
$$

$$
A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \frac{1}{2}[-2 a \cos \theta]^{2} d \theta=2 a^{2} \int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \cos ^{2} \theta d \theta
$$


$r=2$

$r=-2$

$r=4 \sin \theta$

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## Circle in Polar Coordinates

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r=-2 a \cos \theta, \quad \frac{\pi}{2} \leq \theta \leq \frac{3 \pi}{2}
$$

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A=\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \frac{1}{2}[-2 a \cos \theta]^{2} d \theta=2 a^{2} \int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \cos ^{2} \theta d \theta
$$


$r=2$

$r=-2$

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## Circle in Polar Coordinates

$$
r=-2 a \cos \theta, \quad \frac{\pi}{2} \leq \theta \leq \frac{3 \pi}{2}
$$

$$
\begin{aligned}
A & =\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \frac{1}{2}[-2 a \cos \theta]^{2} d \theta=2 a^{2} \int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \cos ^{2} \theta d \theta \\
& =2 a^{2} \int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}}\left(\frac{1}{2}+\frac{1}{2} \cos 2 \theta\right) d \theta=2 a^{2} \int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \frac{1}{2} d \theta=2 a^{2} \cdot \frac{\pi}{2}=\pi a^{2}
\end{aligned}
$$


$r=2$

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## Circle in Polar Coordinates

$$
r=-2 a \cos \theta, \quad \frac{\pi}{2} \leq \theta \leq \frac{3 \pi}{2}
$$

$$
\begin{aligned}
A & =\int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \frac{1}{2}[-2 a \cos \theta]^{2} d \theta=2 a^{2} \int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \cos ^{2} \theta d \theta \\
& =2 a^{2} \int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}}\left(\frac{1}{2}+\frac{1}{2} \cos 2 \theta\right) d \theta=2 a^{2} \int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \frac{1}{2} d \theta=2 a^{2} \cdot \frac{\pi}{2}=\pi a^{2}
\end{aligned}
$$

## Area of a Lemniscate (Ribbon): $A=a^{2}$




## Ribbon

Sketch $r^{2}=a^{2} \cos 2 \theta$ in 4 stages: $\left[0, \frac{\pi}{4}\right],\left[\frac{\pi}{4}, \frac{\pi}{2}\right],\left[\frac{\pi}{2}, \frac{3 \pi}{4}\right],\left[\frac{3 \pi}{4}, \pi\right]$


## Area of a Lemniscate (Ribbon): $A=a^{2}$




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Sketch $r^{2}=a^{2} \cos 2 \theta$ in 4 stages: $\left[0, \frac{\pi}{4}\right],\left[\frac{\pi}{4}, \frac{\pi}{2}\right],\left[\frac{\pi}{2}, \frac{3 \pi}{4}\right],\left[\frac{3 \pi}{4}, \pi\right]$

$$
A=4 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=4 \int_{0}^{\frac{\pi}{4}} \frac{1}{2}\left(a^{2} \cos 2 \theta\right) d \theta
$$

## Area of a Lemniscate (Ribbon): $A=a^{2}$




## Ribbon

Sketch $r^{2}=a^{2} \cos 2 \theta$ in 4 stages: $\left[0, \frac{\pi}{4}\right],\left[\frac{\pi}{4}, \frac{\pi}{2}\right],\left[\frac{\pi}{2}, \frac{3 \pi}{4}\right],\left[\frac{3 \pi}{4}, \pi\right]$

$$
A=4 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=4 \int_{0}^{\frac{\pi}{4}} \frac{1}{2}\left(a^{2} \cos 2 \theta\right) d \theta=2 a^{2}\left[\frac{1}{2} \sin 2 \theta\right]
$$

## Area of a Lemniscate (Ribbon): $A=a^{2}$




## Ribbon

Sketch $r^{2}=a^{2} \cos 2 \theta$ in 4 stages: $\left[0, \frac{\pi}{4}\right],\left[\frac{\pi}{4}, \frac{\pi}{2}\right],\left[\frac{\pi}{2}, \frac{3 \pi}{4}\right],\left[\frac{3 \pi}{4}, \pi\right]$

$$
A=4 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=4 \int_{0}^{\frac{\pi}{4}} \frac{1}{2}\left(a^{2} \cos 2 \theta\right) d \theta=2 a^{2}\left[\frac{1}{2} \sin 2 \theta\right]_{0}^{\frac{\pi}{4}}=a^{2}
$$

## Area of a Lemniscate (Ribbon): $A=a^{2}$


$r^{2}=4 \sin 2 \theta$

$r^{2}=4 \cos 2 \theta$

## Ribbon

Sketch $r^{2}=a^{2} \sin 2 \theta$ in 4 stages: $\left[0, \frac{\pi}{4}\right],\left[\frac{\pi}{4}, \frac{\pi}{2}\right],\left[\frac{\pi}{2}, \frac{3 \pi}{4}\right],\left[\frac{3 \pi}{4}, \pi\right]$


## Area of a Lemniscate (Ribbon): $A=a^{2}$


$r^{2}=4 \sin 2 \theta$

$r^{2}=4 \cos 2 \theta$

## Ribbon

Sketch $r^{2}=a^{2} \sin 2 \theta$ in 4 stages: $\left[0, \frac{\pi}{4}\right],\left[\frac{\pi}{4}, \frac{\pi}{2}\right],\left[\frac{\pi}{2}, \frac{3 \pi}{4}\right],\left[\frac{3 \pi}{4}, \pi\right]$

$$
A=4 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=4 \int_{0}^{\pi} \frac{1}{2}\left(a^{2} \sin 2 \theta\right) d \theta
$$

## Area of a Lemniscate (Ribbon): $A=a^{2}$


$r^{2}=4 \sin 2 \theta$

$r^{2}=4 \cos 2 \theta$

## Ribbon

Sketch $r^{2}=a^{2} \sin 2 \theta$ in 4 stages: $\left[0, \frac{\pi}{4}\right],\left[\frac{\pi}{4}, \frac{\pi}{2}\right],\left[\frac{\pi}{2}, \frac{3 \pi}{4}\right],\left[\frac{3 \pi}{4}, \pi\right]$

$$
\begin{equation*}
A=4 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=4 \int_{0}^{\frac{\pi}{4}} \frac{1}{2}\left(a^{2} \sin 2 \theta\right) d \theta \tag{0}
\end{equation*}
$$

## Area of a Lemniscate (Ribbon): $A=a^{2}$


$r^{2}=4 \sin 2 \theta$

$r^{2}=4 \cos 2 \theta$

## Ribbon

Sketch $r^{2}=a^{2} \sin 2 \theta$ in 4 stages: $\left[0, \frac{\pi}{4}\right],\left[\frac{\pi}{4}, \frac{\pi}{2}\right],\left[\frac{\pi}{2}, \frac{3 \pi}{4}\right],\left[\frac{3 \pi}{4}, \pi\right]$

$$
A=4 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=4 \int_{0}^{\frac{\pi}{4}} \frac{1}{2}\left(a^{2} \sin 2 \theta\right) d \theta=2 a^{2}\left[-\frac{1}{2} \cos 2 \theta\right]_{0}^{\frac{\pi}{4}}=a^{2}
$$

## Quiz

## Quiz

1. $r=2 a \sin \theta$ is $a$
(a) line,
(b) circle,
(c) lemniscate.
2. $r^{2}=a^{2} \sin 2 \theta$ is a (a) line,
(b) circle,
(c) lemniscate.

## Area of a Flower: $A=\frac{\pi}{4}, \frac{\pi}{2}$




Flower
Sketch $r=\sin 3 \theta$ in 6 stages: $\left[0, \frac{\pi}{6}\right],\left[\frac{\pi}{6}, \frac{\pi}{3}\right], \cdots,\left[\frac{2 \pi}{3}, \frac{5 \pi}{6}\right],\left[\frac{5 \pi}{6}, \pi\right]$


## Area of a Flower: $A=\frac{\pi}{4}, \frac{\pi}{2}$




Flower
Sketch $r=\sin 3 \theta$ in 6 stages: $\left[0, \frac{\pi}{6}\right],\left[\frac{\pi}{6}, \frac{\pi}{3}\right], \cdots,\left[\frac{2 \pi}{3}, \frac{5 \pi}{6}\right],\left[\frac{5 \pi}{6}, \pi\right]$

$$
A=\sigma \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=\sigma \int_{0}^{\sigma} \frac{1}{2}[\sin 3 \theta]^{2} d \theta
$$

## Area of a Flower: $A=\frac{\pi}{4}, \frac{\pi}{2}$




## Flower

Sketch $r=\sin 3 \theta$ in 6 stages: $\left[0, \frac{\pi}{6}\right],\left[\frac{\pi}{6}, \frac{\pi}{3}\right], \cdots,\left[\frac{2 \pi}{3}, \frac{5 \pi}{6}\right],\left[\frac{5 \pi}{6}, \pi\right]$

$$
\begin{aligned}
A & =6 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=6 \int_{0}^{\frac{\pi}{6}} \frac{1}{2}[\sin 3 \theta]^{2} d \theta \\
& =3 \int_{0}^{\sigma}\left(\frac{1}{2}-\frac{1}{2} \cos 6 \theta\right) d \theta=\frac{3}{2}\left[\theta-\frac{1}{6} \sin \right.
\end{aligned}
$$

## Area of a Flower: $A=\frac{\pi}{4}, \frac{\pi}{2}$




## Flower

Sketch $r=\sin 3 \theta$ in 6 stages: $\left[0, \frac{\pi}{6}\right],\left[\frac{\pi}{6}, \frac{\pi}{3}\right], \cdots,\left[\frac{2 \pi}{3}, \frac{5 \pi}{6}\right],\left[\frac{5 \pi}{6}, \pi\right]$

$$
\begin{aligned}
A & =6 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=6 \int_{0}^{\frac{\pi}{6}} \frac{1}{2}[\sin 3 \theta]^{2} d \theta \\
& =3 \int_{0}^{\frac{\pi}{6}}\left(\frac{1}{2}-\frac{1}{2} \cos 6 \theta\right) d \theta=\frac{3}{2}\left[\theta-\frac{1}{6} \sin 6 \theta\right]_{0}^{\sigma}=\frac{\pi}{4}
\end{aligned}
$$

## Area of a Flower: $A=\frac{\pi}{4}, \frac{\pi}{2}$




## Flower

Sketch $r=\sin 3 \theta$ in 6 stages: $\left[0, \frac{\pi}{6}\right],\left[\frac{\pi}{6}, \frac{\pi}{3}\right], \cdots,\left[\frac{2 \pi}{3}, \frac{5 \pi}{6}\right],\left[\frac{5 \pi}{6}, \pi\right]$

$$
\begin{aligned}
A & =6 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=6 \int_{0}^{\frac{\pi}{6}} \frac{1}{2}[\sin 3 \theta]^{2} d \theta \\
& =3 \int_{0}^{\frac{\pi}{6}}\left(\frac{1}{2}-\frac{1}{2} \cos 6 \theta\right) d \theta=\frac{3}{2}\left[\theta-\frac{1}{6} \sin 6 \theta\right]_{0}^{\frac{\pi}{6}}=\frac{\pi}{4}
\end{aligned}
$$

## Area of a Flower: $A=\frac{\pi}{4}, \frac{\pi}{2}$




Flower
Sketch $r=\cos 4 \theta$ in 16 stages: $\left[0, \frac{\pi}{8}\right],\left[\frac{\pi}{8}, \frac{\pi}{4}\right], \cdots,\left[\frac{15 \pi}{8}, 2 \pi\right]$

$$
A=16 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=16 \int_{0} \frac{1}{2}[\cos 4 \theta]^{2} d \theta
$$

## Area of a Flower: $A=\frac{\pi}{4}, \frac{\pi}{2}$




Flower
Sketch $r=\cos 4 \theta$ in 16 stages: $\left[0, \frac{\pi}{8}\right],\left[\frac{\pi}{8}, \frac{\pi}{4}\right], \cdots,\left[\frac{15 \pi}{8}, 2 \pi\right]$

$$
\begin{aligned}
A & =16 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=16 \int_{0}^{\frac{\pi}{8}} \frac{1}{2}[\cos 4 \theta]^{2} d \theta \\
& =8 \int_{0}^{\frac{\pi}{8}}\left(\frac{1}{2}+\frac{1}{2} \cos 8 \theta\right) d \theta=4\left[\theta+\frac{1}{8} \sin 8 \theta\right.
\end{aligned}
$$

## Area of a Flower: $A=\frac{\pi}{4}, \frac{\pi}{2}$




Flower
Sketch $r=\cos 4 \theta$ in 16 stages: $\left[0, \frac{\pi}{8}\right],\left[\frac{\pi}{8}, \frac{\pi}{4}\right], \cdots,\left[\frac{15 \pi}{8}, 2 \pi\right]$

$$
\begin{aligned}
A & =16 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=16 \int_{0}^{\frac{\pi}{8}} \frac{1}{2}[\cos 4 \theta]^{2} d \theta \\
& =8 \int_{0}^{\frac{\pi}{8}}\left(\frac{1}{2}+\frac{1}{2} \cos 8 \theta\right) d \theta=4\left[\theta+\frac{1}{8} \sin 8 \theta\right.
\end{aligned}
$$

## Area of a Flower: $A=\frac{\pi}{4}, \frac{\pi}{2}$




## Flower

Sketch $r=\cos 4 \theta$ in 16 stages: $\left[0, \frac{\pi}{8}\right],\left[\frac{\pi}{8}, \frac{\pi}{4}\right], \cdots,\left[\frac{15 \pi}{8}, 2 \pi\right]$

$$
\begin{aligned}
A & =16 \int_{\alpha}^{\beta} \frac{1}{2}[\rho(\theta)]^{2} d \theta=16 \int_{0}^{\frac{\pi}{8}} \frac{1}{2}[\cos 4 \theta]^{2} d \theta \\
& =8 \int_{0}^{\frac{\pi}{8}}\left(\frac{1}{2}+\frac{1}{2} \cos 8 \theta\right) d \theta=4\left[\theta+\frac{1}{8} \sin 8 \theta\right]_{0}^{\frac{\pi}{8}}=\frac{\pi}{2}
\end{aligned}
$$

## Limaçons (Snails): $r=a+\cos \theta, a \geq 1$


$r=3+\cos \theta$ convex
limaçon

$r=\frac{3}{2}+\cos \theta$ limaçon with a dimple


$$
r=1+\cos \theta
$$ cardioid


$r=\frac{1}{2}+\cos \theta$ limaçon with an inner loop

## Flower

Sketch $r=a+\cos \theta, a \geq 1$, in 2 stages: $[0, \pi],[\pi, 2 \pi]$

## Limaçons (Snails): $r=a+\cos \theta, a \geq 1$


$r=3+\cos \theta$ convex
limaçon

$r=\frac{3}{2}+\cos \theta$ limaçon with a dimple

$r=1+\cos \theta$ cardioid

$r=\frac{1}{2}+\cos \theta$ limaçon with an inner loop

## Flower

Sketch $r=a+\cos \theta, a \geq 1$, in 2 stages: $[0, \pi],[\pi, 2 \pi]$

$$
A=2 \int_{0}^{\pi} \frac{1}{2}[a+\cos \theta]^{2} d \theta=\int\left(a^{2}+2 a \cos \theta+\cos ^{2} \theta\right) d \theta
$$

## Limaçons (Snails): $r=a+\cos \theta, a \geq 1$


$r=3+\cos \theta$ convex
limaçon

$r=\frac{3}{2}+\cos \theta$ limaçon with a dimple

$r=1+\cos \theta$ cardioid

$r=\frac{1}{2}+\cos \theta$ limaçon with an inner loop

## Flower

Sketch $r=a+\cos \theta, a \geq 1$, in 2 stages: $[0, \pi],[\pi, 2 \pi]$

$$
A=2 \int_{0}^{\pi} \frac{1}{2}[a+\cos \theta]^{2} d \theta=\int_{0}^{\pi}\left(a^{2}+2 a \cos \theta+\cos ^{2} \theta\right) d \theta
$$

## Limaçons (Snails): $r=a+\cos \theta, a \geq 1$


$r=3+\cos \theta$ convex
limaçon

$r=\frac{3}{2}+\cos \theta$ limaçon with a dimple

$r=1+\cos \theta$ cardioid

$r=\frac{1}{2}+\cos \theta$ limaçon with an inner loop

## Flower

Sketch $r=a+\cos \theta, a \geq 1$, in 2 stages: $[0, \pi],[\pi, 2 \pi]$

$$
A=2 \int_{0}^{\pi} \frac{1}{2}[a+\cos \theta]^{2} d \theta=\int_{0}^{\pi}\left(a^{2}+2 a \cos \theta+\cos ^{2} \theta\right) d \theta
$$

## Limaçons (Snails): $r=a+\cos \theta, a \geq 1$


$r=3+\cos \theta$ convex
limaçon

$r=\frac{3}{2}+\cos \theta$ limaçon with a dimple

$r=1+\cos \theta$ cardioid

$r=\frac{1}{2}+\cos \theta$ limaçon with an inner loop

## Flower

Sketch $r=a+\cos \theta, a \geq 1$, in 2 stages: $[0, \pi],[\pi, 2 \pi]$

$$
\begin{aligned}
& A=2 \int_{0}^{\pi} \frac{1}{2}[a+\cos \theta]^{2} d \theta=\int_{0}^{\pi}\left(a^{2}+2 a \cos \theta+\cos ^{2} \theta\right) d \theta \\
& =\int_{0}^{\pi}\left(a^{2}+2 a \cos \theta+\frac{1}{2}+\frac{1}{2} \cos 2 \theta\right) d \theta=\left[\left(a^{2}+\frac{1}{2}\right) \theta\right]_{0}^{\pi}=
\end{aligned}
$$

## Limaçons (Snails): $r=a+\cos \theta, a \geq 1$


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limaçon


$$
r=\frac{3}{2}+\cos \theta
$$

limaçon
with a dimple

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& =\int_{0}^{\pi}\left(a^{2}+2 a \cos \theta+\frac{1}{2}+\frac{1}{2} \cos 2 \theta\right) d \theta=\left[\left(a^{2}+\frac{1}{2}\right) \theta\right]_{0}^{\pi}=\left(a^{2}+\frac{1}{2}\right) \pi
\end{aligned}
$$

## Limaçon (Snail): $r=1-2 \cos \theta$



$0 \leq \theta \leq \frac{1}{3} \pi$


## Area between the Inner and Outer Loops

- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
- $A=A_{\text {outer }}-A_{\text {inner }}$

Area Within Outer Loop: $A_{\text {outer }}$
$A_{\text {outer }}$

$0 \leq \theta \leq \pi$
 $[1-2 \cos \theta]^{2} d \theta$

## Limaçon (Snail): $r=1-2 \cos \theta$



$0 \leq \theta \leq \frac{1}{3} \pi$


Area between the Inner and Outer Loops

- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
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- $A=A_{\text {outer }}-A_{\text {inner }}$

Area Within Outer Loop: $A_{\text {outer }}$
$A_{\text {outer }}=2 \int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}[1-2 \cos \theta]^{2} d \theta=\int_{\frac{\pi}{3}}^{\pi}\left(1-4 \cos \theta+4 \cos ^{2} \theta\right) d \theta$

## Limaçon (Snail): $r=1-2 \cos \theta$



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## Area between the Inner and Outer Loops

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## Limaçon (Snail): $r=1-2 \cos \theta$



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## Area between the Inner and Outer Loops

- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
- $A=A_{\text {outer }}-A_{\text {inner }}$

Area Within Outer Loop: $A_{\text {outer }}$

$$
\begin{aligned}
& A_{\text {outer }}=2 \int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}[1-2 \cos \theta]^{2} d \theta=\int_{\frac{\pi}{3}}^{\pi}\left(1-4 \cos \theta+4 \cos ^{2} \theta\right) d \theta \\
& =\int_{\frac{\pi}{3}}^{\pi}(1-4 \cos \theta+2+2 \cos 2 \theta) d \theta=[3 \theta-4 \sin \theta+\sin 2 \theta]=
\end{aligned}
$$

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$0 \leq \theta \leq \frac{1}{3} \pi$


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- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
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Area Within Outer Loop: $A_{\text {outer }}$

$$
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\end{aligned}
$$

## Limaçon (Snail): $r=1-2 \cos \theta$



$0 \leq \theta \leq \frac{1}{3} \pi$


$0 \leq \theta \leq \pi$


## Area between the Inner and Outer Loops

- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
- $A=A_{\text {outer }}-A_{\text {inner }}=\left(2 \pi+\frac{3}{2} \sqrt{3}\right)-A_{\text {inner }}$

Area Within Outer Loop: $A_{\text {outer }}$

$$
\begin{aligned}
& A_{\text {outer }}=2 \int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}[1-2 \cos \theta]^{2} d \theta=\int_{\frac{\pi}{3}}^{\pi}\left(1-4 \cos \theta+4 \cos ^{2} \theta\right) d \theta \\
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\end{aligned}
$$

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- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
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## Area Within Inner Loop: $A_{\text {inner }}$

## Limaçon (Snail): $r=1-2 \cos \theta$



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## Area between the Inner and Outer Loops

- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
- $A=A_{\text {outer }}-A_{\text {inner }}=\left(2 \pi+\frac{3}{2} \sqrt{3}\right)-A_{\text {inner }}$

Area Within Inner Loop: $A_{\text {inner }}$
$A_{\text {inner }}=2 \int \frac{1}{2}[1-2 \cos \theta]^{2} d \theta$

## Limaçon (Snail): $r=1-2 \cos \theta$



$0 \leq \theta \leq \frac{1}{3} \pi$


$0 \leq \theta \leq \pi$


## Area between the Inner and Outer Loops

- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
- $A=A_{\text {outer }}-A_{\text {inner }}=\left(2 \pi+\frac{3}{2} \sqrt{3}\right)-A_{\text {inner }}$

Area Within Inner Loop: $A_{\text {inner }}$

$$
A_{\text {inner }}=2 \int_{0}^{\frac{\pi}{3}} \frac{1}{2}[1-2 \cos \theta]^{2} d \theta=\int_{0}^{\frac{\pi}{3}}\left(1-4 \cos \theta+4 \cos ^{2} \theta\right) d \theta
$$

## Limaçon (Snail): $r=1-2 \cos \theta$



$0 \leq \theta \leq \frac{1}{3} \pi$


$0 \leq \theta \leq \pi$


## Area between the Inner and Outer Loops

- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
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Area Within Inner Loop: $A_{\text {inner }}$

$$
A_{\text {inner }}=2 \int_{0}^{\frac{\pi}{3}} \frac{1}{2}[1-2 \cos \theta]^{2} d \theta=\int_{0}^{\frac{\pi}{3}}\left(1-4 \cos \theta+4 \cos ^{2} \theta\right) d \theta
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$0 \leq \theta \leq \frac{1}{3} \pi$


$0 \leq \theta \leq \pi$


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$$
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& A_{\text {inner }}=2 \int_{0}^{\frac{\pi}{3}} \frac{1}{2}[1-2 \cos \theta]^{2} d \theta=\int_{0}^{\frac{\pi}{3}}\left(1-4 \cos \theta+4 \cos ^{2} \theta\right) d \theta \\
& =\int_{0}^{\frac{\pi}{3}}(1-4 \cos \theta+2+2 \cos 2 \theta) d \theta=[3 \theta-4 \sin \theta+\sin 2 \theta]_{0}^{\frac{\pi}{3}}=
\end{aligned}
$$

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$0 \leq \theta \leq \frac{1}{3} \pi$


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Area Within Inner Loop: $A_{\text {inner }}$

$$
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$0 \leq \theta \leq \frac{1}{3} \pi$


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## Area between the Inner and Outer Loops

- Sketch in 4 stages: $\left[0, \frac{1}{3} \pi\right],\left[\frac{1}{3} \pi, \pi\right],\left[\pi, \frac{5}{3} \pi\right],\left[\frac{5}{3} \pi, 2 \pi\right]$.
- $A=A_{\text {outer }}-A_{\text {inner }}=\left(2 \pi+\frac{3}{2} \sqrt{3}\right)-\left(\pi-\frac{3}{2} \sqrt{3}\right)=\pi+3 \sqrt{3}$

Area Within Inner Loop: $A_{\text {inner }}$

$$
\begin{aligned}
& A_{\text {inner }}=2 \int_{0}^{\frac{\pi}{3}} \frac{1}{2}[1-2 \cos \theta]^{2} d \theta=\int_{0}^{\frac{\pi}{3}}\left(1-4 \cos \theta+4 \cos ^{2} \theta\right) d \theta \\
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\end{aligned}
$$

## Area between Polar Curves



Area between $r=\rho_{1}(\theta)$ and $r=\rho_{2}(\theta)$
area of $\Omega=\int^{b_{2}} \frac{1}{2}\left[\rho_{2}(\theta)\right]^{2} d \theta$

## Area between Polar Curves



Area between $r=\rho_{1}(\theta)$ and $r=\rho_{2}(\theta)$

$$
\text { area of } \begin{align*}
& \Omega=\int_{\alpha_{2}}^{\beta_{2}} \frac{1}{2}\left[\rho_{2}(\theta)\right]^{2} d \theta \\
&-\int_{\alpha_{1}}^{\beta_{1}} \frac{1}{2}\left[\rho_{1}(\theta)\right]^{2} d \theta \\
&=\int_{\alpha}^{\beta} \frac{1}{2}\left(\left[\rho_{2}(\theta)\right]^{2}-\left[\rho_{1}(\theta)\right]^{2}\right)
\end{align*}
$$

$\square$

## Area between Polar Curves



$$
\begin{aligned}
& \text { Area between } r=\rho_{1}(\theta) \text { and } r=\rho_{2}(\theta) \\
& \text { area of } \Omega=\int_{\alpha_{2}}^{\beta_{2}} \frac{1}{2}\left[\rho_{2}(\theta)\right]^{2} d \theta \\
& \quad-\int_{\alpha_{1}}^{\beta_{1}} \frac{1}{2}\left[\rho_{1}(\theta)\right]^{2} d \theta \\
& =\int_{\alpha}^{\beta} \frac{1}{2}\left(\left[\rho_{2}(\theta)\right]^{2}-\left[\rho_{1}(\theta)\right]^{2}\right) d \theta \\
& \quad \text { if } \quad \alpha_{1}=\alpha_{2}=\alpha, \quad \beta_{1}=\beta_{2}=\beta
\end{aligned}
$$

Remark
Extra care is needed to determine the intervals of $\theta$ values (e.g, $\left[\alpha_{1}, \beta_{1}\right]$ and $\left[\alpha_{2}, \beta_{2}\right]$ ) over which the outer and inner boundaries of the region are traced out

## Area between Polar Curves



Area between $r=\rho_{1}(\theta)$ and $r=\rho_{2}(\theta)$

$$
\begin{aligned}
& \text { area of } \Omega=\int_{\alpha_{2}}^{\beta_{2}} \frac{1}{2}\left[\rho_{2}(\theta)\right]^{2} d \theta \\
& \\
& -\int_{\alpha_{1}}^{\beta_{1}} \frac{1}{2}\left[\rho_{1}(\theta)\right]^{2} d \theta \\
& =\int_{\alpha}^{\beta} \frac{1}{2}\left(\left[\rho_{2}(\theta)\right]^{2}-\left[\rho_{1}(\theta)\right]^{2}\right) d \theta \\
& \text { if } \quad \alpha_{1}=\alpha_{2}=\alpha, \quad \beta_{1}=\beta_{2}=\beta .
\end{aligned}
$$

## Remark

Extra care is needed to determine the intervals of $\theta$ values (e.g, [ $\alpha_{1}, \beta_{1}$ ] and $\left[\alpha_{2}, \beta_{2}\right]$ ) over which the outer and inner boundaries of the region are traced out.


## Area between $r=2 \cos \theta$ and $r=1$

- The two intersection points:



## Area between Circles: $r=2 \cos \theta$ and $r=1$



## Area between $r=2 \cos \theta$ and $r=1$

- The two intersection points:

$$
\begin{aligned}
& 2 \cos \theta=1 \quad \Rightarrow \quad \cos \theta=\frac{1}{2} \\
& \frac{\pi}{3}, \frac{5 \pi}{3} \quad \Rightarrow \quad[\alpha, \beta]=\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]
\end{aligned} \Rightarrow
$$

- By symmetry,



## Area between Circles: $r=2 \cos \theta$ and $r=1$



## Area between $r=2 \cos \theta$ and $r=1$

- The two intersection points:

$$
\begin{aligned}
& 2 \cos \theta=1 \quad \Rightarrow \quad \cos \theta=\frac{1}{2} \quad \Rightarrow \\
& \frac{\pi}{3}, \frac{5 \pi}{3} \quad \Rightarrow \quad[\alpha, \beta]=\left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \\
& \text { By symmetry, } \\
& \quad \text { area of } \Omega
\end{aligned}
$$



## Area between Circles: $r=2 \cos \theta$ and $r=1$



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\end{aligned} \Rightarrow
$$

- By symmetry,
area of $\Omega$
$=\int_{\alpha}^{\beta} \frac{1}{2}\left(\left[\rho_{2}(\theta)\right]^{2}-\left[\rho_{1}(\theta)\right]^{2}\right) d \theta$



## Area between Circles: $r=2 \cos \theta$ and $r=1$



## Area between $r=2 \cos \theta$ and $r=1$

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\end{aligned} \Rightarrow
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area of $\Omega$

$$
\begin{aligned}
& =\int_{\alpha}^{\beta} \frac{1}{2}\left(\left[\rho_{2}(\theta)\right]^{2}-\left[\rho_{1}(\theta)\right]^{2}\right) d \theta \\
& =2 \int_{0}^{\frac{\pi}{3}} \frac{1}{2}\left([2 \cos \theta]^{2}-[1]^{2}\right) d \theta
\end{aligned}
$$

## Area between Circles: $r=2 \cos \theta$ and $r=1$



## Area between $r=2 \cos \theta$ and $r=1$

- The two intersection points:

$$
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$$

- By symmetry,
area of $\Omega$

$$
\begin{aligned}
& =\int_{\alpha}^{\beta} \frac{1}{2}\left(\left[\rho_{2}(\theta)\right]^{2}-\left[\rho_{1}(\theta)\right]^{2}\right) d \theta \\
& =2 \int_{0}^{\frac{\pi}{3}} \frac{1}{2}\left([2 \cos \theta]^{2}-[1]^{2}\right) d \theta \\
& =\int_{0}^{\frac{\pi}{3}}(1+2 \cos 2 \theta) d \theta
\end{aligned}
$$

## Area between Circles: $r=2 \cos \theta$ and $r=1$



## Area between $r=2 \cos \theta$ and $r=1$

- The two intersection points:

$$
\begin{aligned}
& 2 \cos \theta=1 \\
& \frac{\pi}{3}, \frac{5 \pi}{3} \quad \Rightarrow \quad \cos \theta=\frac{1}{2} \quad \Rightarrow
\end{aligned} \quad \Rightarrow
$$

- By symmetry,
area of $\Omega$

$$
\begin{aligned}
& =\int_{\alpha}^{\beta} \frac{1}{2}\left(\left[\rho_{2}(\theta)\right]^{2}-\left[\rho_{1}(\theta)\right]^{2}\right) d \theta \\
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& =\int_{0}^{\frac{\pi}{3}}(1+2 \cos 2 \theta) d \theta \\
& =[\theta+\sin 2 \theta]_{0}^{\frac{\pi}{3}}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}
\end{aligned}
$$

Quiz
3. area of $r=2 a \sin \theta$ is :
(a) $\pi a^{2}$,
(b) $\frac{1}{2} \pi a^{2}, \quad$ (c) $a^{2}$.
4. area of $r^{2}=a^{2} \sin 2 \theta$ is :
(a) $\pi a^{2}$,
(b) $\frac{1}{2} \pi a^{2}$,
(c) $a^{2}$.

## Area outside Circle $r=2 \sin \theta$ and inside flower $r=2 \sin 2 \theta$



- The three intersection points: $2 \sin 2 \theta=2 \sin \theta \Rightarrow 2 \sin \theta \cos \theta=\sin \theta \Rightarrow$ $\sin \theta(2 \cos \theta-1)=0 \Rightarrow \sin \theta=0$ or $\cos \theta=\frac{1}{2} \Rightarrow 0$, or $\frac{\pi}{3}, \frac{5 \pi}{3} \Rightarrow[\alpha, \beta]=\left[0, \frac{\pi}{3}\right]$ and $[\alpha, \beta]=\left[\frac{5 \pi}{3}, \pi\right]$


## Area outside Circle $r=2 \sin \theta$ and inside flower $r=2 \sin 2 \theta$



- The three intersection points:
$2 \sin 2 \theta=2 \sin \theta \Rightarrow 2 \sin \theta \cos \theta=\sin \theta \Rightarrow$ $\sin \theta(2 \cos \theta-1)=0 \Rightarrow \sin \theta=0$ or $\cos \theta=\frac{1}{2} \Rightarrow 0$, or $\frac{\pi}{3}, \frac{5 \pi}{3} \Rightarrow[\alpha, \beta]=\left[0, \frac{\pi}{3}\right]$ and $[\alpha, \beta]=\left[\frac{5 \pi}{3}, \pi\right]$


## Area outside Circle $r=2 \sin \theta$ and inside flower $r=2 \sin 2 \theta$



- The three intersection points: $2 \sin 2 \theta=2 \sin \theta \Rightarrow 2 \sin \theta \cos \theta=\sin \theta \Rightarrow$ $\sin \theta(2 \cos \theta-1)=0 \Rightarrow \sin \theta=0$ or $\cos \theta=\frac{1}{2} \Rightarrow 0$, or $\frac{\pi}{3}, \frac{5 \pi}{3} \Rightarrow[\alpha, \beta]=\left[0, \frac{\pi}{3}\right]$ and $[\alpha, \beta]=\left[\frac{5 \pi}{3}, \pi\right]$
- By symmetry,


## $A_{1}$



## Area outside Circle $r=2 \sin \theta$ and inside flower $r=2 \sin 2 \theta$



- The three intersection points: $2 \sin 2 \theta=2 \sin \theta \Rightarrow 2 \sin \theta \cos \theta=\sin \theta \Rightarrow$ $\sin \theta(2 \cos \theta-1)=0 \Rightarrow \sin \theta=0$ or $\cos \theta=\frac{1}{2} \Rightarrow 0$, or $\frac{\pi}{3}, \frac{5 \pi}{3} \Rightarrow[\alpha, \beta]=\left[0, \frac{\pi}{3}\right]$ and $[\alpha, \beta]=\left[\frac{5 \pi}{3}, \pi\right]$
- By symmetry,

$$
A_{1}=\int_{\alpha}^{\beta} \frac{1}{2}\left(\left[\rho_{2}(\theta)\right]^{2}-\left[\rho_{1}(\theta)\right]^{2}\right) d \theta
$$

## Area outside Circle $r=2 \sin \theta$ and inside flower $r=2 \sin 2 \theta$



- The three intersection points: $2 \sin 2 \theta=2 \sin \theta \Rightarrow 2 \sin \theta \cos \theta=\sin \theta \Rightarrow$ $\sin \theta(2 \cos \theta-1)=0 \Rightarrow \sin \theta=0$ or $\cos \theta=\frac{1}{2} \Rightarrow 0$, or $\frac{\pi}{3}, \frac{5 \pi}{3} \Rightarrow[\alpha, \beta]=\left[0, \frac{\pi}{3}\right]$ and $[\alpha, \beta]=\left[\frac{5 \pi}{3}, \pi\right]$
- By symmetry,

$$
\begin{aligned}
& A_{1}=\int_{\alpha}^{\beta} \frac{1}{2}\left(\left[\rho_{2}(\theta)\right]^{2}-\left[\rho_{1}(\theta)\right]^{2}\right) d \theta \\
& =\int_{0}^{\frac{\pi}{3}} \frac{1}{2}\left([2 \sin 2 \theta]^{2}-[2 \sin \theta]^{2}\right) d \theta \\
& =\int_{0}^{\frac{\pi}{3}}(\cos 2 \theta-\cos 4 \theta) d \theta
\end{aligned}
$$

## Area outside Circle $r=2 \sin \theta$ and inside flower $r=2 \sin 2 \theta$

$$
\sin \theta(2 \cos \theta-1)=0 \Rightarrow \sin \theta=0 \text { or }
$$

$$
\cos \theta=\frac{1}{2} \Rightarrow 0, \text { or } \frac{\pi}{3}, \frac{5 \pi}{3} \Rightarrow[\alpha, \beta]=\left[0, \frac{\pi}{3}\right]
$$



- The three intersection points:

$$
2 \sin 2 \theta=2 \sin \theta \Rightarrow 2 \sin \theta \cos \theta=\sin \theta \Rightarrow
$$

$$
\text { and }[\alpha, \beta]=\left[\frac{5 \pi}{3}, \pi\right]
$$

- By symmetry,

$$
\begin{aligned}
& A_{1}=\int_{\alpha}^{\beta} \frac{1}{2}\left(\left[\rho_{2}(\theta)\right]^{2}-\left[\rho_{1}(\theta)\right]^{2}\right) d \theta \\
& =\int_{0}^{\frac{\pi}{3}} \frac{1}{2}\left([2 \sin 2 \theta]^{2}-[2 \sin \theta]^{2}\right) d \theta \\
& =\int_{0}^{\frac{\pi}{3}}(\cos 2 \theta-\cos 4 \theta) d \theta \\
& =\left[\frac{1}{2} \sin 2 \theta-\frac{1}{4} \sin 4 \theta\right]_{0}^{\frac{\pi}{3}}=\frac{3 \sqrt{3}}{8}
\end{aligned}
$$

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- The two intersection points:
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$$
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A & =\int_{0}^{\frac{\pi}{6}} \frac{1}{2}[2 \sin \theta]^{2} d \theta \\
& +\int_{\frac{\pi}{6}}^{\frac{5 \pi}{6}} \frac{1}{2}\left[\frac{3}{2}-\sin \theta\right]^{2} d \theta
\end{aligned}
$$

## Area between Circle $r=2 \sin \theta$ and Limaçon $r=\frac{3}{2}-\sin \theta$



- The two intersection points:
$2 \sin \theta=\frac{3}{2}-\sin \theta \Rightarrow \sin \theta=\frac{1}{2} \Rightarrow \frac{\pi}{6}, \frac{5 \pi}{6}$
- The area can be represented as follows:

$$
\begin{aligned}
A & =\int_{0}^{\frac{\pi}{6}} \frac{1}{2}[2 \sin \theta]^{2} d \theta \\
& +\int_{\frac{\pi}{6}}^{\frac{5 \pi}{6}} \frac{1}{2}\left[\frac{3}{2}-\sin \theta\right]^{2} d \theta \\
& +\int_{\frac{5 \pi}{6}}^{\pi} \frac{1}{2}[2 \sin \theta]^{2} d \theta
\end{aligned}
$$

## Area between Circle $r=2 \sin \theta$ and Limaçon $r=\frac{3}{2}-\sin \theta$



- The two intersection points:
$2 \sin \theta=\frac{3}{2}-\sin \theta \Rightarrow \sin \theta=\frac{1}{2} \Rightarrow \frac{\pi}{6}, \frac{5 \pi}{6}$
- The area can be represented as follows:

$$
\begin{aligned}
A & =\int_{0}^{\frac{\pi}{6}} \frac{1}{2}[2 \sin \theta]^{2} d \theta \\
& +\int_{\frac{\pi}{6}}^{\frac{5 \pi}{6}} \frac{1}{2}\left[\frac{3}{2}-\sin \theta\right]^{2} d \theta \\
& +\int_{\frac{5 \pi}{6}}^{\pi} \frac{1}{2}[2 \sin \theta]^{2} d \theta \\
& =\cdots=\frac{5}{4} \pi-\frac{15}{8} \sqrt{3}
\end{aligned}
$$

- Area of a Polar Region
- Basic Polar Area
- Circles
- Ribbons
- Flowers
- Limaçons
- Area between Polar Curves
- Between Polar Curves
- Between Circles
- Between Circle and Flower
- Between Circle and Limaçon

