Lecture 14

Section 9.6 Curves Given Parametrically

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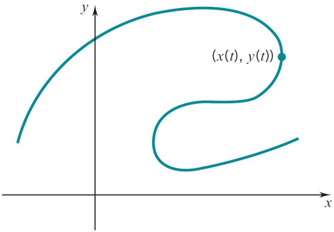
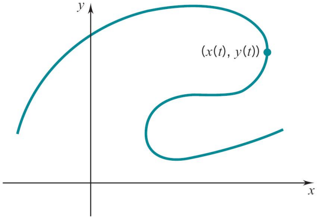
http://math.uh.edu/∼jiwenhe/Math1432



Parametrized curve



Parametrized curve

A parametrized Curve is a path in the xy-plane traced out by the point $(x(t), y(t))$ as the parameter t ranges over an interval I.

$$C = \{(x(t), y(t)) : t \in I\}$$

Examples

- The graph of a function $y = f(x)$, $x \in I$, is a curve C that is parametrized by
$$x(t) = t, \quad y(t) = f(t), \quad t \in I.$$
- The graph of a polar equation $r = \rho(\theta)$, $\theta \in I$, is a curve C that is parametrized by the functions
$$x(t) = r\cos t = \rho(t)\cos t, \quad y(t) = r\sin t = \rho(t)\sin t, \quad t \in I.$$

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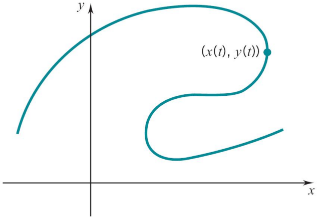

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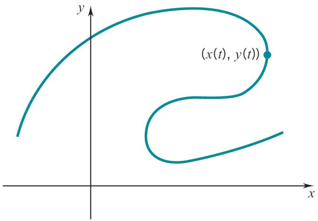

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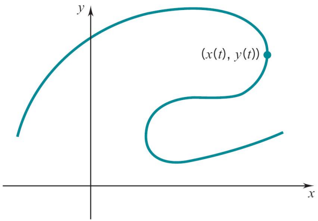

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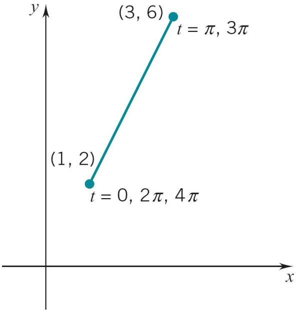

Line Segment: $y = 2x,\ x \in [1,3]$

- Set $x(t) = t$, then $y(t) = 2t$, $t \in [1,3]$
- Set $x(t) = t + 1$, then $y(t) = 2t + 2$, $t \in [0,2]$
- Set $x(t) = 3 - t$, then $y(t) = 6 - 2t$, $t \in [0,2]$
- Set $x(t) = 3 - 4t$, then $y(t) = 6 - 8t$, $t \in [0,1/2]$
- Set $x(t) = 2 - \cos t$, then $y(t) = 4 - 2\cos t$, $t \in [0,4\pi]$

We parametrize the line segment in different ways and interpret each parametrization as the motion of a particle with the parameter t being time.

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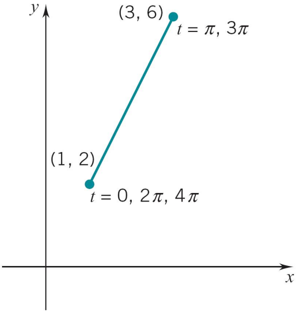

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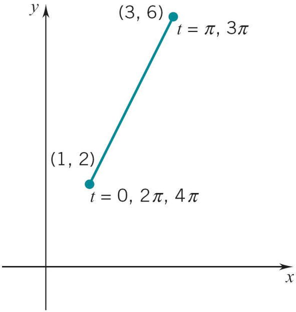

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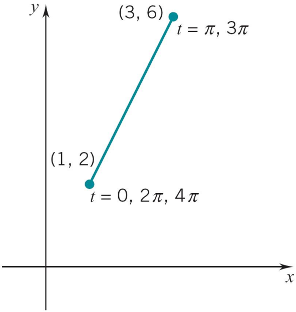

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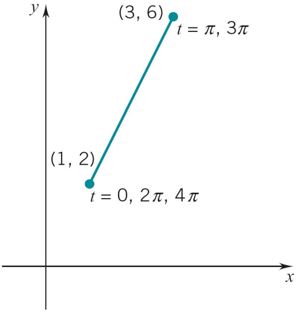

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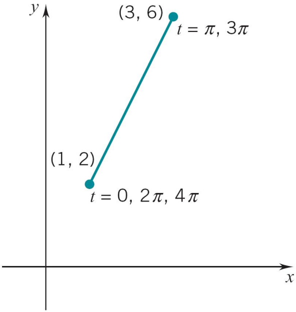

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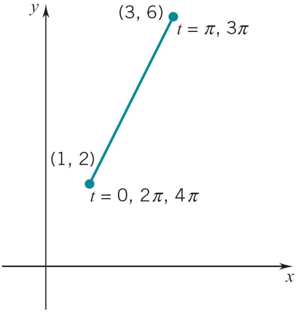

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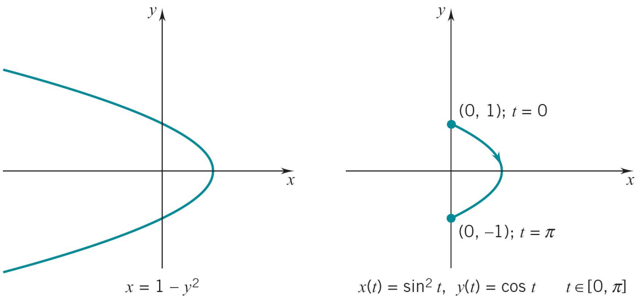
Example: Parabola



Parabola Arc: $x = 1 - y^2$, $-1 \le y \le 1$

- Set $y(t) = t$, then $x(t) = 1 - t^2$, $t \in [-1, 1] \Rightarrow$ changing the domain to all real t gives us the whole parabola.
- Set $y(t) = \cos t$, then $x(t) = 1 - \cos^2 t$, $t \in [0, \pi] \Rightarrow$ changing the domain to all real t does not give us any more of the parabola.

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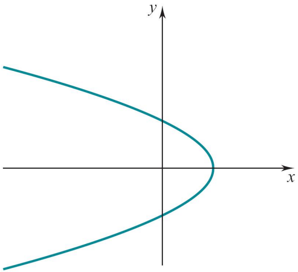
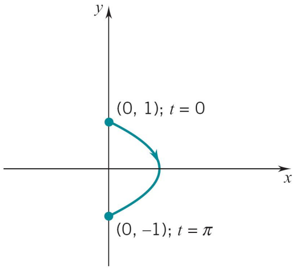


$x = 1 - y^2$



$x(t) = \sin^2 t,\ \ y(t) = \cos t \qquad t \in [0, \pi]$

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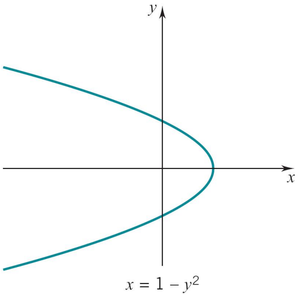

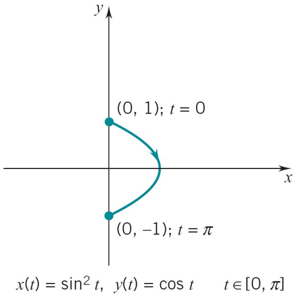

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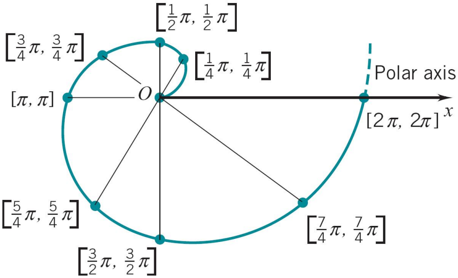

$r = \theta, \quad \theta \geq 0$

spiral of Archimedes

Spiral of Archimedes: $r = \theta$, $\theta \geq 0$

- The curve is a nonending spiral. Here it is shown in detail from $\theta = 0$ to $\theta = 2\pi$.
- The parametric representation is

$$x(t) = t\cos t, \quad y(t) = t\sin t, \quad t \geq 0.$$

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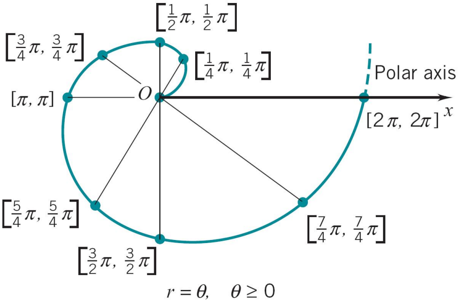

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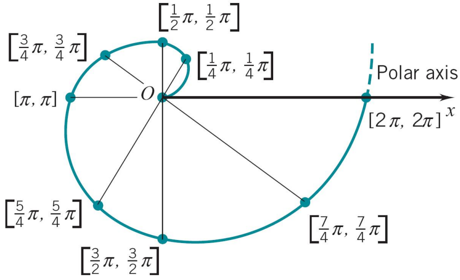

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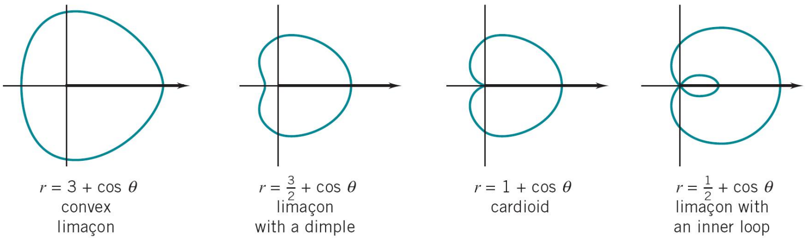
Example: Limaçons

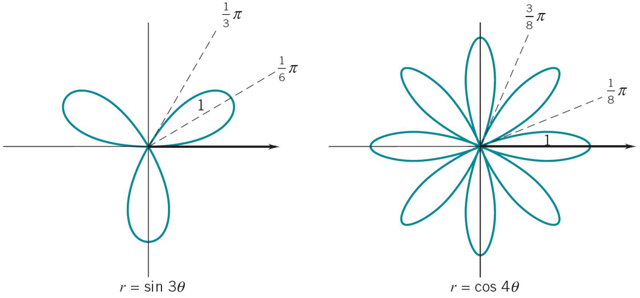


Limaçons (Snails): $r = a + b\cos\theta$

The parametric representation is

$$x(t) = \big(a + b\cos t\big)\cos t, \quad y(t) = \big(a + b\cos t\big)\sin t, \quad t \in [0, 2\pi].$$

Example: Petal Curves



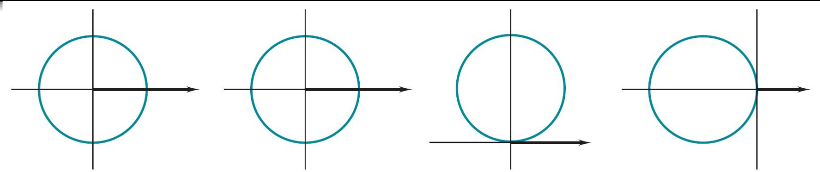
Petal Curves (Flowers): $r = a\cos n\theta$, $r = a\sin n\theta$

The parametric representations are

$$x(t) = \big(a\cos(nt)\big)\cos t, \quad y(t) = \big(a\cos(nt)\big)\sin t, \qquad t \in [0, 2\pi].$$

$$x(t) = \big(a\sin(nt)\big)\cos t, \quad y(t) = \big(a\sin(nt)\big)\sin t, \qquad t \in [0, 2\pi].$$

Circles: $C = \{P : d(P, O) = |a|\}$



Center O at $(0,0) \Rightarrow x^2 + y^2 = a^2 \Rightarrow r = a$

$$\Rightarrow \quad t \in [0, 2\pi], \quad x(t) = a\cos t, \quad y(t) = a \sin t$$

Center O at $(0,a) \Rightarrow x^2 + (y-a)^2 = a^2 \Rightarrow r = 2a\sin\theta$

$$\Rightarrow \quad t \in [0, \pi], \quad \begin{cases} x(t) = 2a\sin t\cos t = a \sin 2t, \\ y(t) = 2a \sin t \sin t = a(1-\cos 2t). \end{cases}$$

Another parametric representation is by translation

$$\Rightarrow \quad t \in [0, 2\pi], \quad x(t) = a\cos t, \quad y(t) = a\sin t + a$$

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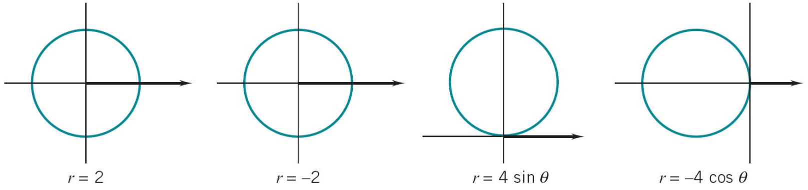

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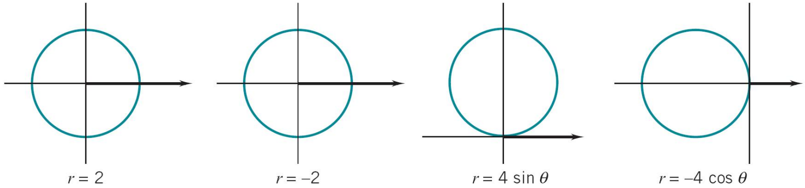

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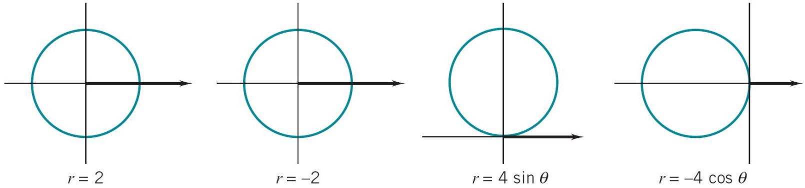

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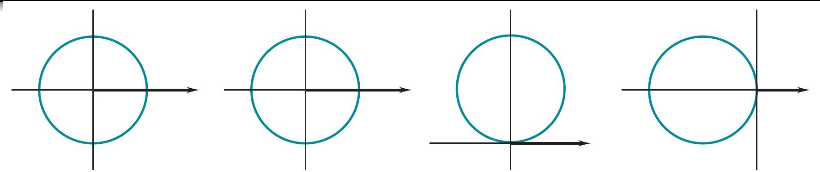

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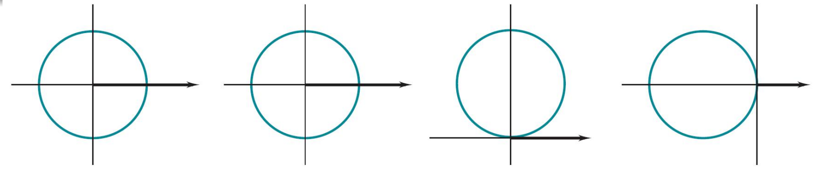

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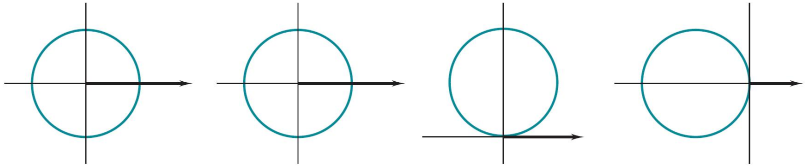

Center O at $(0,0) \Rightarrow x^2 + y^2 = a^2 \Rightarrow r = a$

$$\Rightarrow \quad t \in [0, 2\pi], \quad x(t) = a\cos t, \quad y(t) = a\sin t$$

Center O at $(0,a) \Rightarrow x^2 + (y-a)^2 = a^2 \Rightarrow r = 2a\sin\theta$

$$\Rightarrow \quad t \in [0, \pi], \quad \begin{cases} x(t) = 2a\sin t\cos t = a\sin 2t, \\ y(t) = 2a\sin t\sin t = a(1 - \cos 2t). \end{cases}$$

Another parametric representation is by translation

$$\Rightarrow \quad t \in [0, 2\pi], \quad x(t) = a\cos t, \quad y(t) = a\sin t + a$$

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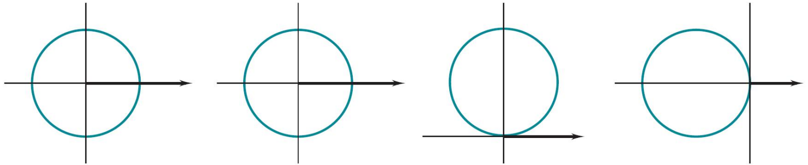

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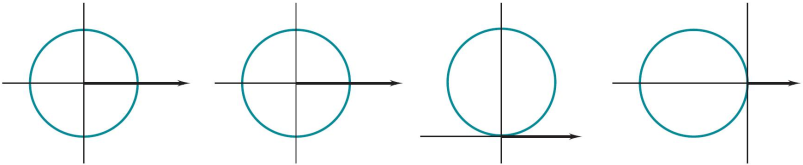

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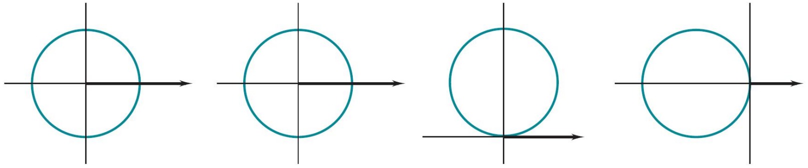
Center O at $(a,0) \Rightarrow (x-a)^2 + y^2 = a^2 \Rightarrow r = 2a\cos\theta$

$$\Rightarrow \quad t \in [\frac{\pi}{2}, \frac{3\pi}{2}], \quad \begin{cases} x(t) = 2a\cos t\cos t = a(1+\cos 2t), \\ y(t) = 2a\cos t\sin t = a\sin 2t. \end{cases}$$

Another parametric representation is by translation

$$\Rightarrow \quad t \in [0, 2\pi], \quad x(t) = a\cos t + a, \quad y(t) = a\sin t.$$

Circles: $C = \{P : d(P, O) = |a|\}$



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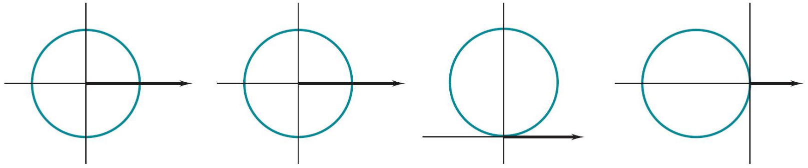

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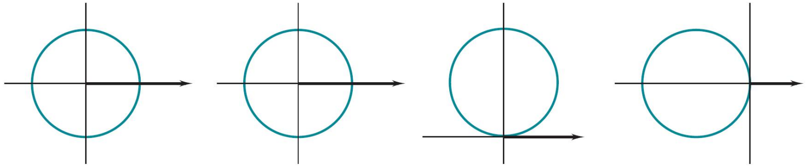

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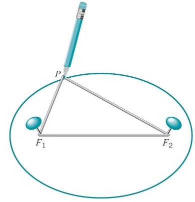
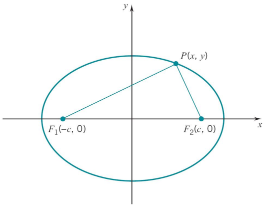
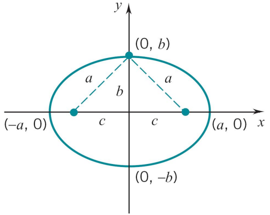
Ellipses



$d(P, F_1) + d(P, F_2) = k$





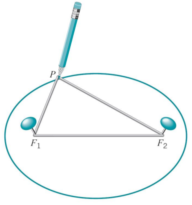
A ellipse is the set of points P in a plane that the sum of whose distances from two fixed points (the foci F_1 and F_2) separated by a distance $2c$ is a given positive constant $2a$.

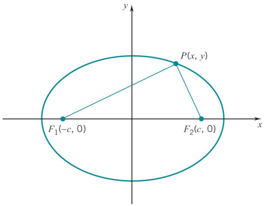
$$E = \left\{ P : |d(P, F_1) + d(P, F_2)| = 2a \right\}$$

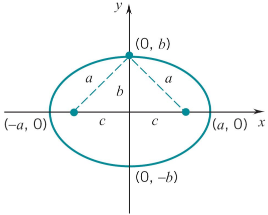
With F_1 at $(-c, 0)$ and F_2 at $(c, 0)$ and setting $b = \sqrt{a^2 - c^2}$,

$$E = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$$

Ellipses







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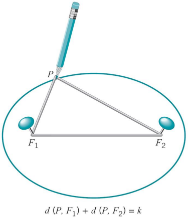
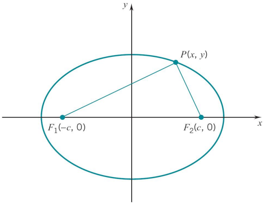
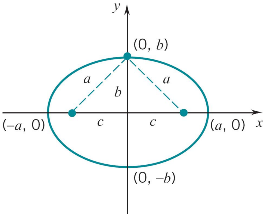
Ellipses







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With F_1 at $(-c, 0)$ and F_2 at $(c, 0)$ and setting $b = \sqrt{a^2 - c^2}$,

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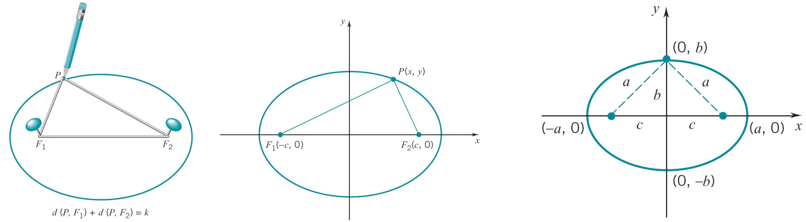
Ellipses: Cosine and Sine

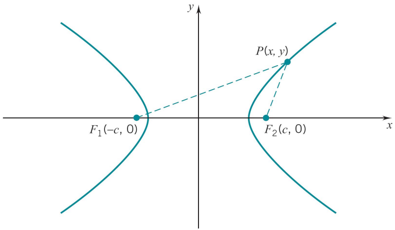


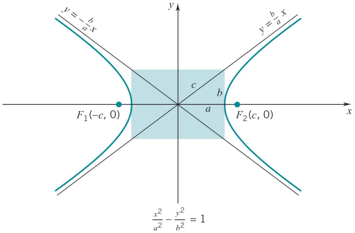
The ellipse can also be given by a simple parametric form analogous to that of a circle, but with the x and y coordinates having different scalings,

$$x = a\cos t, \quad y = b\sin t, \quad t \in (0, 2\pi).$$

Note that $\cos^2 t + \sin^2 t = 1$.

Hyperbolas





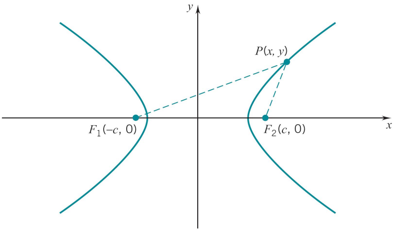
A hyperbola is the set of points P in a plane that the difference of whose distances from two fixed points (the foci F_1 and F_2) separated by a distance $2c$ is a given positive constant $2a$.

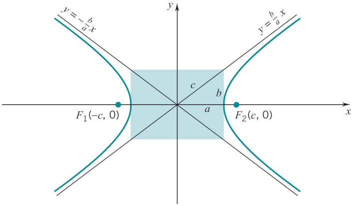
$$H = \{P : |d(P, F_1) - d(P, F_2)| = 2a\}$$

With F_1 at $(-c, 0)$ and F_2 at $(c, 0)$ and setting $b = \sqrt{c^2 - a^2}$, we have

$$H = \left\{(x, y) : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right\}$$

Hyperbolas





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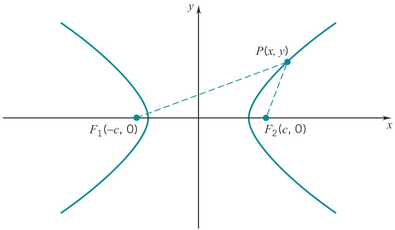
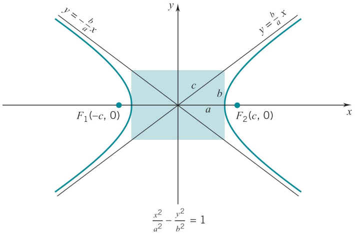
Hyperbolas





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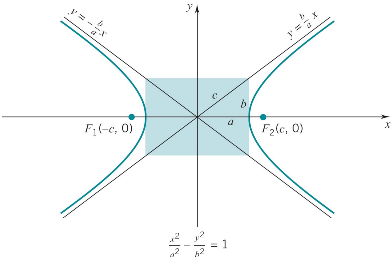
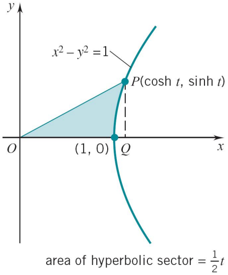
Hyperbolas: Hyperbolic Cosine and Hyperbolic Sine





The right branch of a hyperbola can be parametrized by

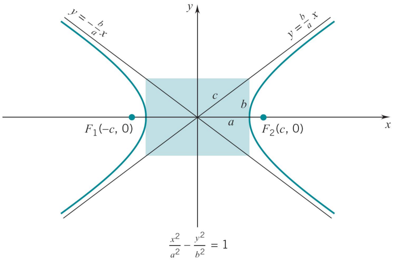
$$x = a\cosh t, \quad y = b\sinh t, \quad t \in (-\infty, \infty).$$

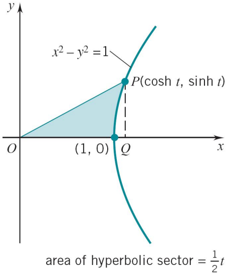
The left branch can be parametrized by

$$x = -a\cosh t, \quad y = b\sinh t, \quad t \in (-\infty, \infty).$$

Note that $\cosh t = \frac{1}{2}(e^t + e^{-t})$, $\sinh t = \frac{1}{2}(e^t - e^{-t})$ and $\cosh^2 t - \sinh^2 t = 1$.

Hyperbolas: Hyperbolic Cosine and Hyperbolic Sine





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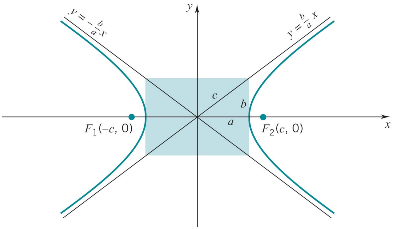
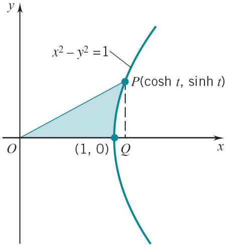
Hyperbolas: Hyperbolic Cosine and Hyperbolic Sine



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



area of hyperbolic sector = $\frac{1}{2}t$

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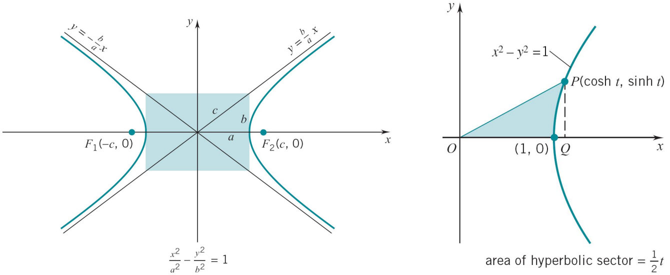
Hyperbolas: Hyperbolic Cosine and Hyperbolic Sine



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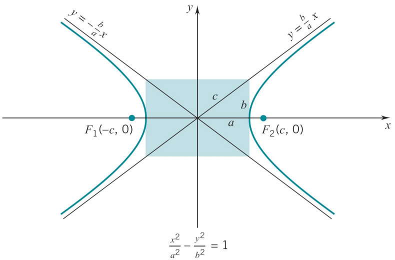
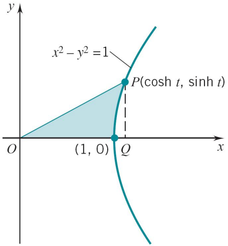
Hyperbolas: Other Parametric Representation





area of hyperbolic sector = $\frac{1}{2}t$

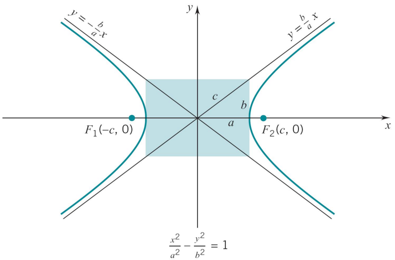
Another parametric representation for the right branch of the hyperbola is

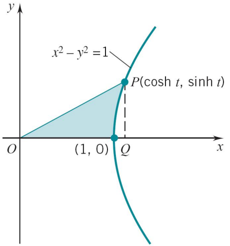
$$x = a\sec t, \quad y = b\tan t, \quad t \in (-\pi/2, \pi/2).$$

Parametric equations for the left branch is

$$x = -a\sec t, \quad y = b\tan t, \quad t \in (-\pi/2, \pi/2).$$

Hyperbolas: Other Parametric Representation





Another parametric representation for the right branch of the hyperbola is

$$x = a \sec t, \quad y = b \tan t, \quad t \in \left(-\pi/2, \pi/2\right).$$

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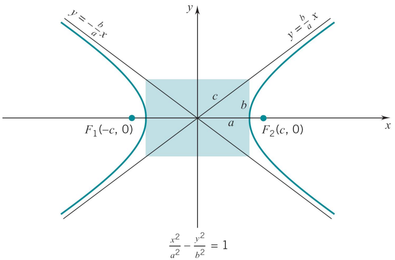
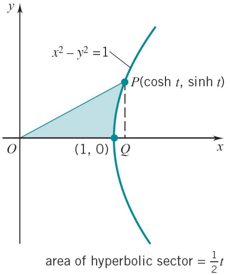
Hyperbolas: Other Parametric Representation





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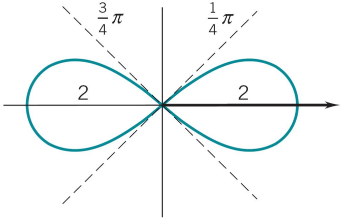
Lemniscates (Ribbons): $r^2 = a^2 \cos 2\theta$



$r^2 = 4 \cos 2\theta$

A lemniscate is the set of points P in a plane that the product of whose distances from two fixed points (the foci F_1 and F_2) a distance $2c$ away is the constant c^2.

$$R = \{P : d(P, F_1) \cdot d(P, F_2)| = c^2\}$$

With F_1 at $(-c, 0)$ and F_2 at $(c, 0)$,

$$(x^2 + y^2)^2 = 2c^2(x^2 - y^2)$$

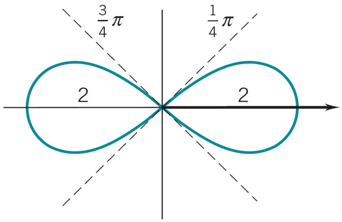
Switching to polar coordinates gives

$$r^2 = 2c^2 \cos 2\theta, \quad \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$$

The parametric equations for the lemniscate with $a^2 = 2c^2$ is

$$x = \frac{a \cos t}{1 + \sin^2 t}, \quad y = \frac{a \sin t \cos t}{1 + \sin^2 t}, \quad t \in (0, 2\pi).$$

Lemniscates (Ribbons): $r^2 = a^2 \cos 2\theta$



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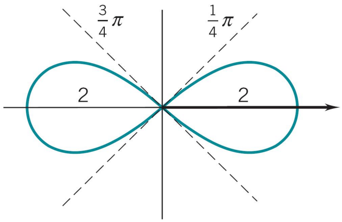
Switching to polar coordinates gives

$$r^2 = 2c^2 \cos 2\theta, \quad \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$$

The parametric equations for the lemniscate with $a^2 = 2c^2$ is

$$x = \frac{a \cos t}{1 + \sin^2 t}, \quad y = \frac{a \sin t \cos t}{1 + \sin^2 t}, \quad t \in (0, 2\pi).$$

Lemniscates (Ribbons): $r^2 = a^2 \cos 2\theta$



$r^2 = 4 \cos 2\theta$

A lemniscate is the set of points P in a plane that the product of whose distances from two fixed points (the foci F_1 and F_2) a distance $2c$ away is the constant c^2.

$$R = \left\{ P : d(P, F_1) \cdot d(P, F_2)| = c^2 \right\}$$

With F_1 at $(-c, 0)$ and F_2 at $(c, 0)$,

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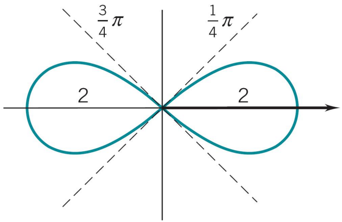

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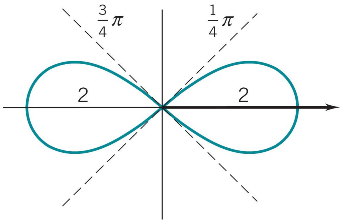

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Outline