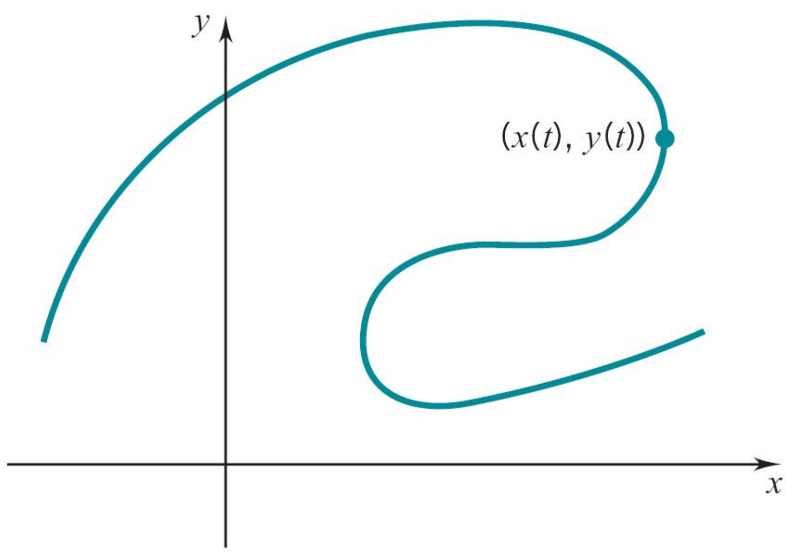
# Lecture 14 Section 9.6 Curves Given Parametrically

**Jiwen He**

# 1 Parametrized curve

## 1.1 Parametrized curve

**Parametrized curve**




**Parametrized curve**

A *parametrized Curve* is a path in the $xy$-plane traced out by the point $(x(t), y(t))$ as the *parameter* $t$ ranges over an interval $I$.

$$C = \{(x(t), y(t)) : t \in I\}$$

*Examples* 1.

- The graph of a *function* $y = f(x)$, $x \in I$, is a curve $C$ that is *parametrized* by

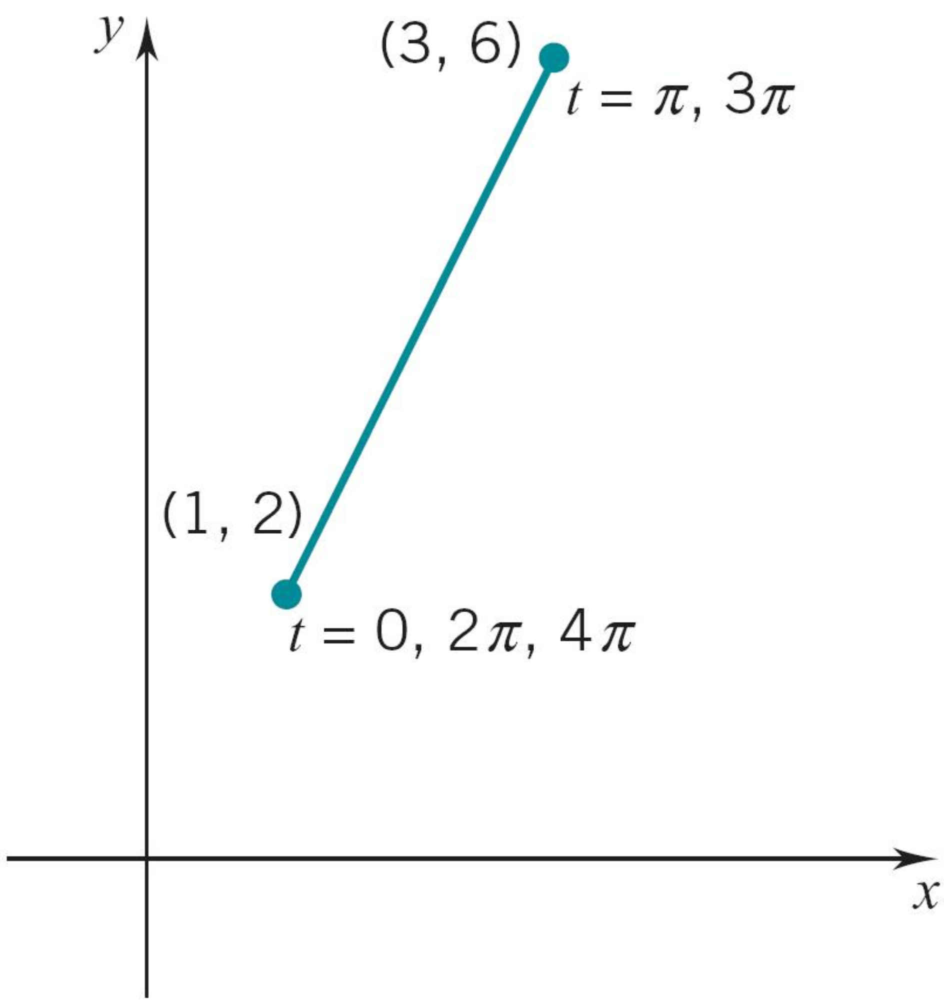
$$x(t) = t, \quad y(t) = f(t), \quad t \in I.$$

- The graph of a *polar equation* $r = \rho(\theta)$, $\theta \in I$, is a curve $C$ that is *parametrized* by the functions

$$x(t) = r\cos t = \rho(t)\cos t, \quad y(t) = r\sin t = \rho(t)\sin t, \quad t \in I.$$

## 1.2 Examples

**Example: Line Segment**



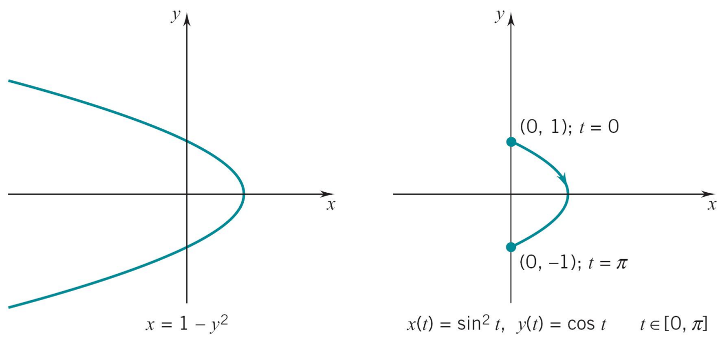

**Line Segment:** $y = 2x$, $x \in [1, 3]$

- Set $x(t) = t$, then $y(t) = 2t$, $t \in [1, 3]$
- Set $x(t) = t + 1$, then $y(t) = 2t + 2$, $t \in [0, 2]$
- Set $x(t) = 3 - t$, then $y(t) = 6 - 2t$, $t \in [0, 2]$
- Set $x(t) = 3 - 4t$, then $y(t) = 6 - 8t$, $t \in [0, 1/2]$

- Set $x(t) = 2 - \cos t$, then $y(t) = 4 - 2\cos t$, $t \in [0, 4\pi]$

We parametrize the line segment in different ways and interpret each parametrization as the motion of a particle with the parameter $t$ being time.

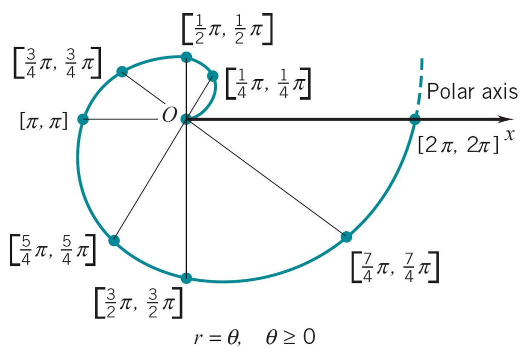
**Example: Parabola**




**Parabola Arc:** $x = 1 - y^2$**,** $-1 \leq y \leq 1$

- Set $y(t) = t$, then $x(t) = 1 - t^2$, $t \in [-1, 1] \Rightarrow$ changing the domain to all real $t$ gives us the whole parabola.
- Set $y(t) = \cos t$, then $x(t) = 1 - \cos^2 t$, $t \in [0, \pi] \Rightarrow$ changing the domain to all real $t$ does not give us any more of the parabola.

**Example: Spiral of Archimedes**

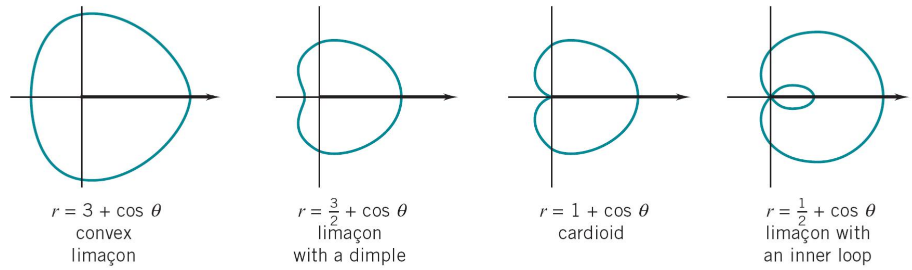



spiral of Archimedes

**Spiral of Archimedes:** $r = \theta$**,** $\theta \geq 0$

- The curve is a nonending spiral. Here it is shown in detail from $\theta = 0$ to $\theta = 2\pi$.
- The parametric representation is
$$x(t) = t\cos t, \quad y(t) = t\sin t, \quad t \geq 0.$$

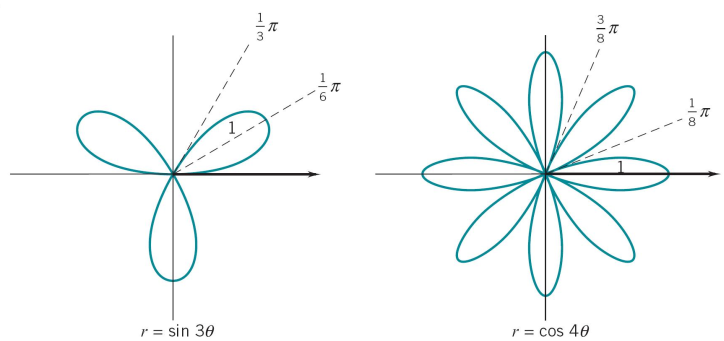
**Example: Limaçons**




**Limaçons (Snails):** $r = a + b\cos\theta$
The parametric representation is

$$x(t) = \big(a + b\cos t\big)\cos t, \quad y(t) = \big(a + b\cos t\big)\sin t, \quad t \in [0, 2\pi].$$

**Example: Petal Curves**




**Petal Curves (Flowers):** $r = a\cos n\theta$**,** $r = a\sin n\theta$
The parametric representations are

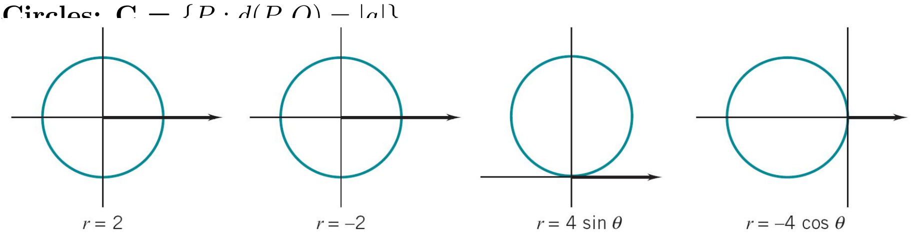
$$x(t) = \big(a\cos(nt)\big)\cos t, \quad y(t) = \big(a\cos(nt)\big)\sin t, \qquad t \in [0, 2\pi].$$

$$x(t) = \big(a\sin(nt)\big)\cos t, \quad y(t) = \big(a\sin(nt)\big)\sin t, \qquad t \in [0, 2\pi].$$

# 2 Locus

## 2.1 Circles

**Circles:** $\mathbf{C} = \{P : d(P, O) = |a|\}$



Center $O$ at $(0,0) \Rightarrow x^2 + y^2 = a^2 \;\; \Rightarrow r = a$

$$\Rightarrow \quad t \in [0, 2\pi], \quad x(t) = a\cos t, \quad y(t) = a\sin t$$

Center $O$ at $(0,a) \Rightarrow x^2 + (y-a)^2 = a^2 \Rightarrow r = 2a\sin\theta$

$$\Rightarrow \quad t \in [0,\pi], \quad \begin{cases} x(t) = 2a\sin t\cos t = a\sin 2t, \\ y(t) = 2a\sin t\sin t = a(1-\cos 2t). \end{cases}$$

Another parametric representation is by translation

$$\Rightarrow \quad t \in [0, 2\pi], \quad x(t) = a\cos t, \quad y(t) = a\sin t + a$$

Center $O$ at $(a,0) \Rightarrow (x-a)^2 + y^2 = a^2 \Rightarrow r = 2a\cos\theta$

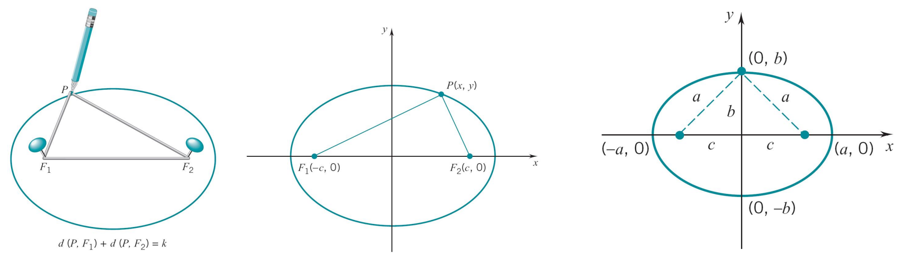
$$\Rightarrow \quad t \in [\frac{\pi}{2}, \frac{3\pi}{2}], \quad \begin{cases} x(t) = 2a\cos t\cos t = a(1+\cos 2t), \\ y(t) = 2a\cos t\sin t = a\sin 2t. \end{cases}$$

Another parametric representation is by translation

$$\Rightarrow \quad t \in [0, 2\pi], \quad x(t) = a\cos t + a, \quad y(t) = a\sin t.$$

## 2.2 Ellipses

**Ellipses**



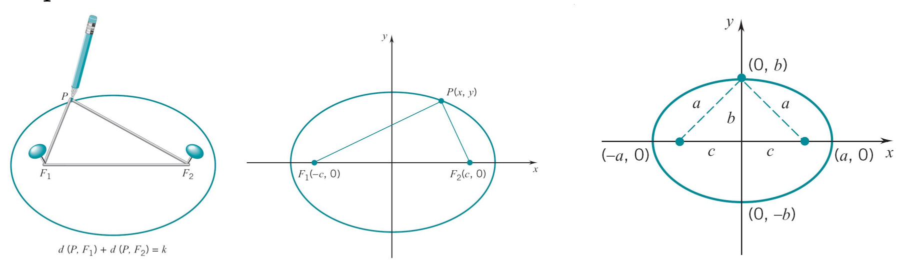

A *ellipse* is the set of points $P$ in a plane that the sum of whose distances from two fixed points (the foci $F_1$ and $F_2$) separated by a distance $2c$ is a given positive constant $2a$.

$$E = \{P : |d(P, F_1) + d(P, F_2)| = 2a\}$$

With $F_1$ at $(-c, 0)$ and $F_2$ at $(c, 0)$ and setting $b = \sqrt{a^2 - c^2}$,

$$E = \left\{(x,y) : \; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$$

**Ellipses: Cosine and Sine**



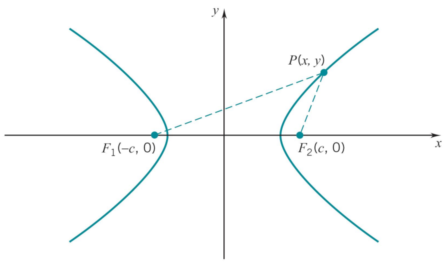

The ellipse can also be given by a simple parametric form analogous to that of a circle, but with the $x$ and $y$ coordinates having different scalings,

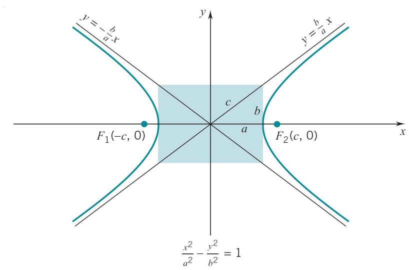
$$x = a\,\cos t, \quad y = b\,\sin t, \quad t \in (0, 2\pi).$$

Note that $\cos^2 t + \sin^2 t = 1$.

## 2.3 Hyperbolas

### Hyperbolas



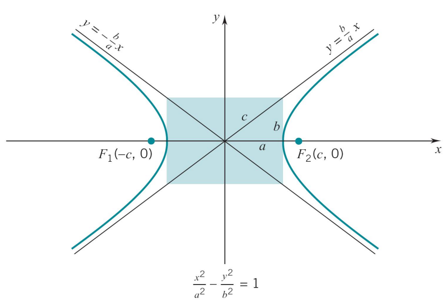

A *hyperbola* is the set of points $P$ in a plane that the difference of whose distances from two fixed points (the foci $F_1$ and $F_2$) separated by a distance $2c$ is a given positive constant $2a$.

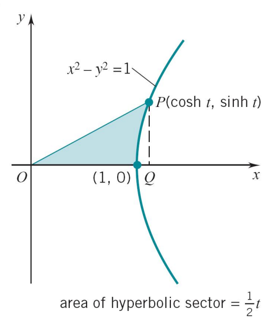
$$H = \left\{ P : |d(P, F_1) - d(P, F_2)| = 2a \right\}$$

With $F_1$ at $(-c, 0)$ and $F_2$ at $(c, 0)$ and setting $b = \sqrt{c^2 - a^2}$, we have

$$H = \left\{ (x, y) : \ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right\}$$

### Hyperbolas: Hyperbolic Cosine and Hyperbolic Sine




The right branch of a hyperbola can be parametrized by

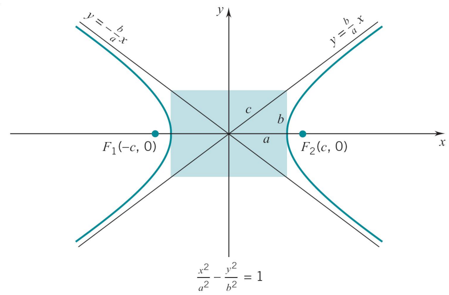
$$x = a \cosh t, \quad y = b \sinh t, \quad t \in (-\infty, \infty).$$

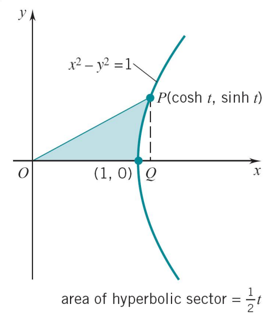
The left branch can be parametrized by

$$x = -a \cosh t, \quad y = b \sinh t, \quad t \in (-\infty, \infty).$$

Note that $\cosh t = \frac{1}{2}\left(e^t + e^{-t}\right)$, $\sinh t = \frac{1}{2}\left(e^t - e^{-t}\right)$ and $\cosh^2 t - \sinh^2 t = 1$.

### Hyperbolas: Other Parametric Representation




Another parametric representation for the right branch of the hyperbola is

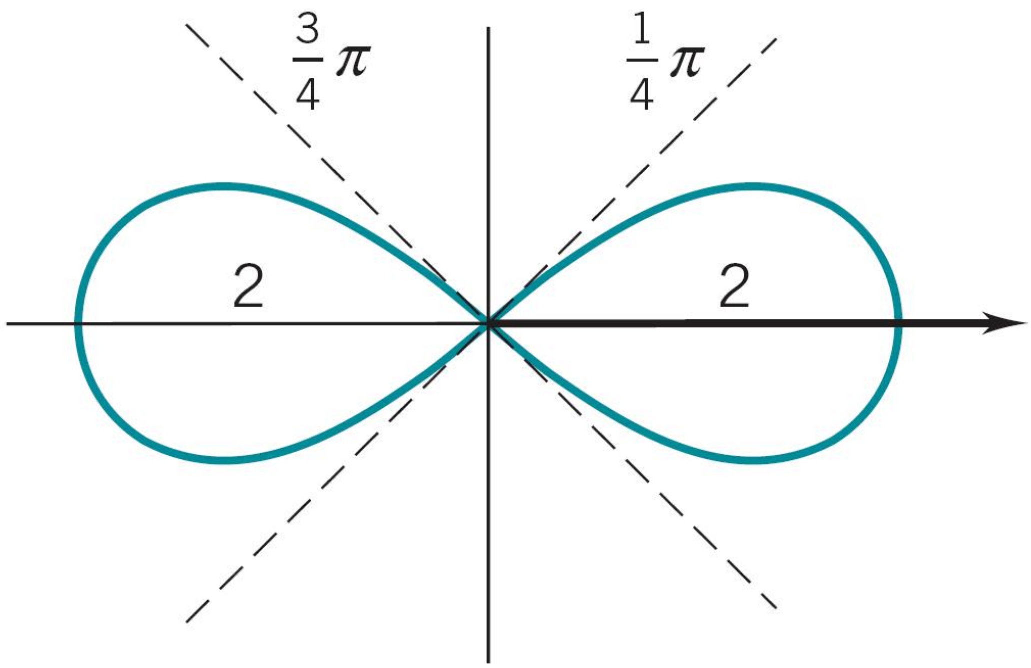
$$x = a\,\sec t, \quad y = b\,\tan t, \quad t \in \big(-\pi/2, \pi/2\big).$$

Parametric equations for the left branch is

$$x = -a\,\sec t, \quad y = b\,\tan t, \quad t \in \big(-\pi/2, \pi/2\big).$$

## 2.4 Lemniscates

**Lemniscates (Ribbons):** $r^2 = a^2 \cos 2\theta$



A *lemniscate* is the set of points $P$ in a plane that the product of whose distances from two fixed points (the foci $F_1$ and $F_2$) a distance $2c$ away is the constant $c^2$.

$$R = \left\{P : d(P, F_1) \cdot d(P, F_2)| = c^2\right\}$$

With $F_1$ at $(-c, 0)$ and $F_2$ at $(c, 0)$,

$$\left(x^2 + y^2\right)^2 = 2c^2\left(x^2 - y^2\right)$$

Switching to polar coordinates gives

$$r^2 = 2c^2 \cos 2\theta,\ \ \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$$

The *parametric equations* for the lemniscate with $a^2 = 2c^2$ is

$$x = \frac{a\ \cos t}{1 + \sin^2 t}, \quad y = \frac{a\ \sin t \cos t}{1 + \sin^2 t}, \quad t \in (0, 2\pi).$$

**Outline**

# Contents