Lecture 15

Section 9.7 Tangents to Curves Given Parametrically

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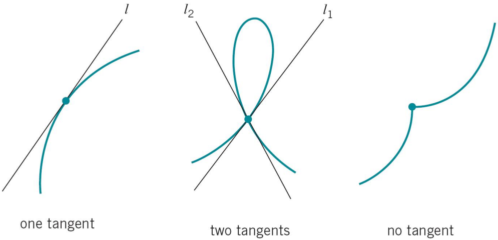
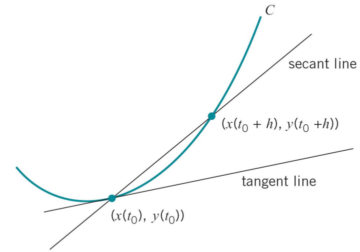


Tangents to Parametrized curves



Tangent line

Let $C = \{(x(t), y(t)) : t \in I\}$.
For a time $t_0 \in I$, assume $x'(t_0) \neq 0$.
The slope of the curve at time t_0 is

$$m(t_0) = \frac{y'(t_0)}{x'(t_0)}$$

The equation of the tangent line is

$$x'(t_0)(y - y_0) - y'(t_0)(x - x_0) = 0$$

Proof.

$$m(t_0) = \lim_{h\to 0} \frac{y(t_0 + h) - y(t_0)}{x(t_0 + h) - x(t_0)} = \frac{\lim_{h\to 0} \dfrac{y(t_0 + h) - y(t_0)}{h}}{\lim_{h\to 0} \dfrac{x(t_0 + h) - x(t_0)}{h}} = \frac{y'(t_0)}{x'(t_0)}$$

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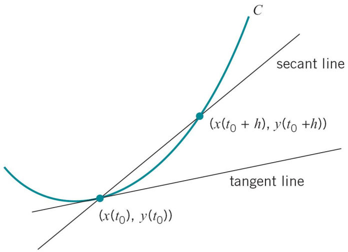

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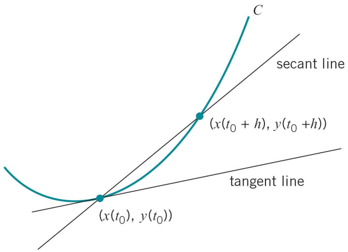

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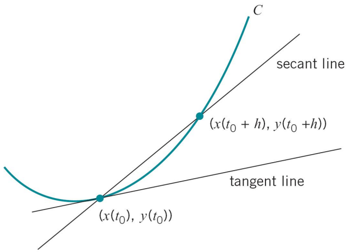

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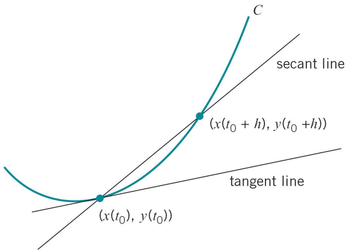

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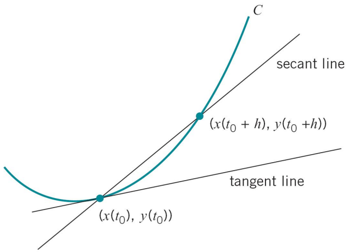

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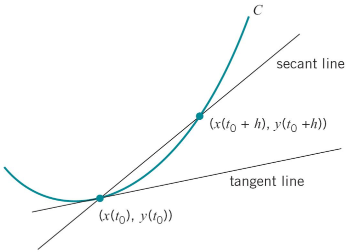

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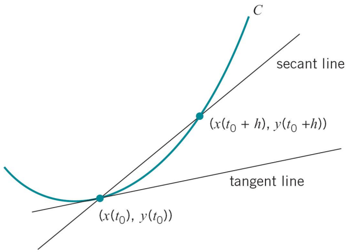

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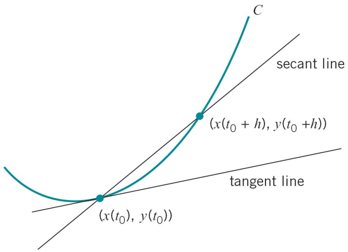

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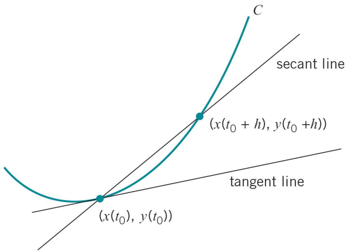

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- The curve has a vertical tangent if $x'(t_0) = 0$
- The curve has a horizontal tangent if $y'(t_0) = 0$

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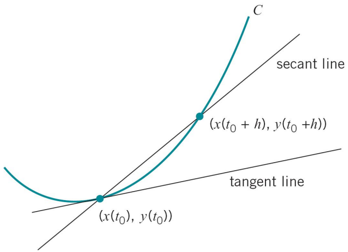

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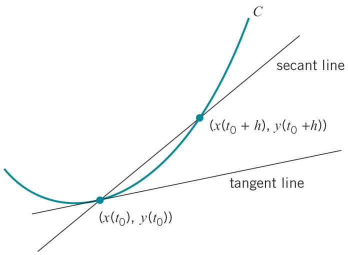

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Example

- The graph of a function $y = f(x)$, $x \in I$, is a curve C that is parametrized by $x(t) = t$, $y(t) = f(t)$, $t \in I$.
- The slope of the curve at time t_0 is

$$m(t_0) = \frac{y'(t_0)}{x'(t_0)} = \frac{dy}{dx}(x_0) = f'(x_0), \quad x_0 = t_0$$

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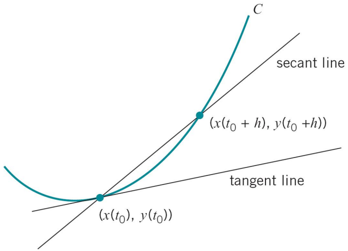

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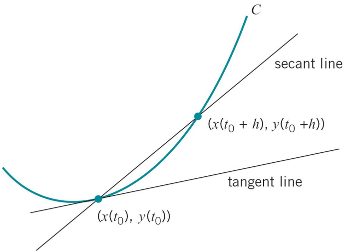
Velocity and Speed Along a Plane Curve



Parametrization by the Motion

- Imaging an object moving along the curve C.
- Let $\mathbf{r}(t) = (x(t), y(t))$ the position of the object at time t.

Velocity and Speed Along a Plane Curve

- The velocity of the object at time t is $\mathbf{v}(t) = \mathbf{r}'(t) = (x'(t), y'(t))$.
- The speed of the object at time t is
$$v(t) = \|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}.$$
- The instantaneous direction of motion gives the unit tangent vector $\mathbf{T}$:
$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)}.$$

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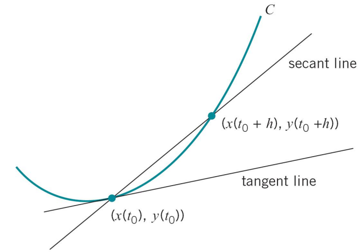

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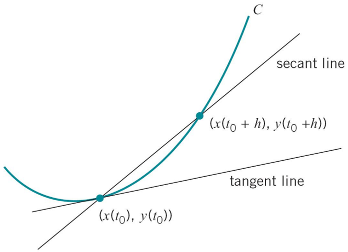

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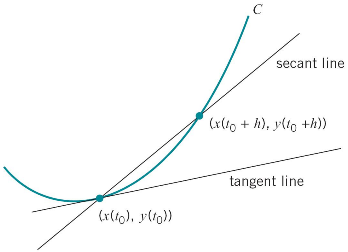

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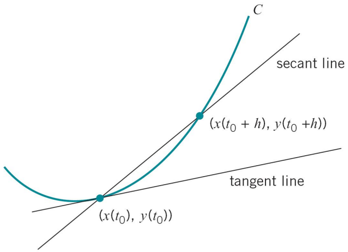

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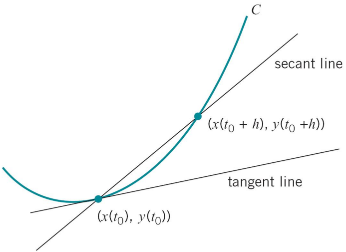

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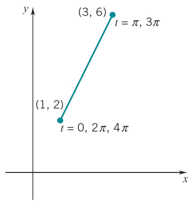

Line Segment: $y = 2x$, $x \in [1, 3]$

- Set $x(t) = t$, then $y(t) = 2t$, $t \in [1, 3]$
- Set $x(t) = 3 - t$, then $y(t) = 6 - 2t$, $t \in [0, 2]$
- Set $x(t) = 2 - \cos t$, then $y(t) = 4 - 2\cos t$, $t \in [0, 4\pi]$.

At time $t \in [1, 3]$:

- The position $\mathbf{r}(t) = (x(t), y(t)) = (t, 2t)$.
- The velocity $\mathbf{v}(t) = (x'(t), y'(t)) = (1, 2)$.
- The speed $v(t) = \|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{5}$.
- The unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{1}{\sqrt{5}}(1, 2)$
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- The tangent line $y = 2x$

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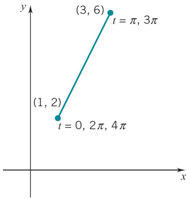

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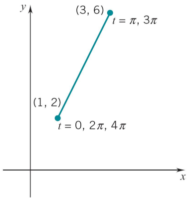

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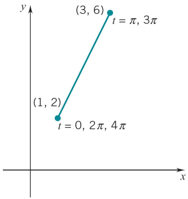

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- The speed $v(t) = \|\mathbf{v}(t)\| = \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2} = \sqrt{5}$.
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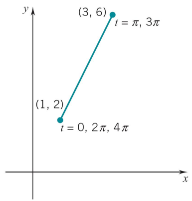
- The slope $m(t) = \frac{y'(t)}{x'(t)} = 2$.
- The tangent line $y = 2x$

Example: Line Segment



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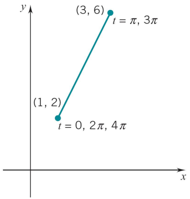

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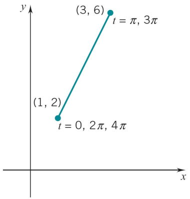

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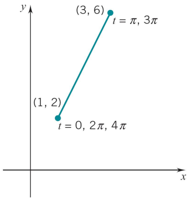

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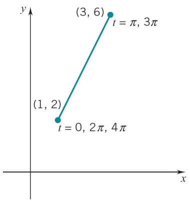

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At time $t \in [0,2]$:

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- The velocity $\mathbf{v}(t) = \big(x'(t), y'(t)\big) = \big(-1, -2\big)$.
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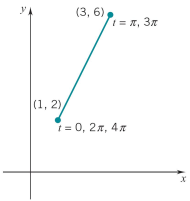

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At time $t \in [0, 2]$:

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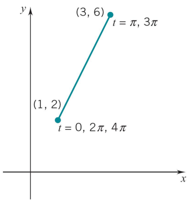

Line Segment: $y = 2x,\ x \in [1, 3]$

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At time $t \in [0, 2]$:

- The position $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big(3 - t, 6 - 2t\big)$.
- The velocity $\mathbf{v}(t) = \big(x'(t), y'(t)\big) = \big(-1, -2\big)$.
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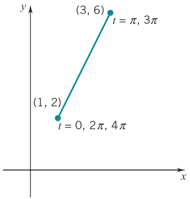

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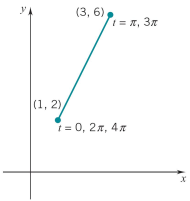

Line Segment: $y = 2x$, $x \in [1, 3]$

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At time $t \in [0, 2]$:

- The position $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big(3 - t, 6 - 2t\big)$.
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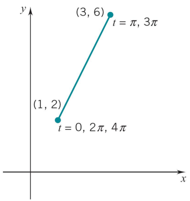

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At time $t \in [0,2]$:

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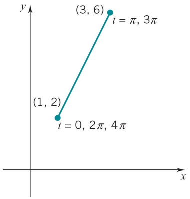

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At time $t \in [0, 4\pi]$:

- The position $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big(2 - \cos t, 4 - 2\cos t\big)$.
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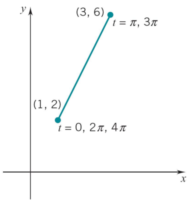

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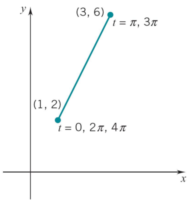

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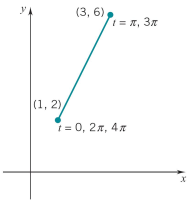

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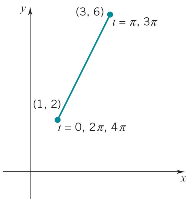

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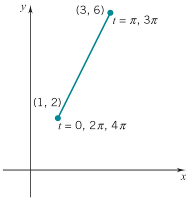

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At time $t \in [0, 4\pi]$:

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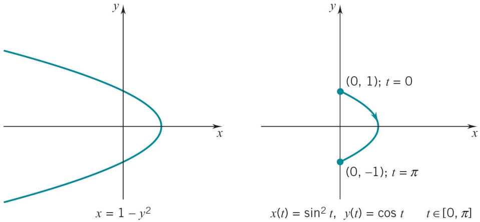
Example: Parabola Arc $x = 1 - y^2$, $-1 \le y \le 1$

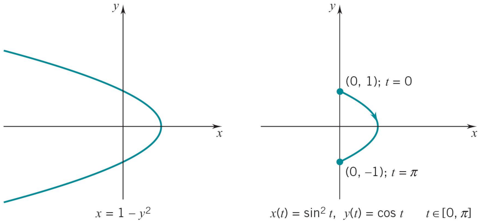


Point of the Vertical Tangent: $\mathbf{r}(t_0) = (1, 0)$

Curve: $x(t) = 1 - t^2$, $y(t) = t$, $t \in [-1, 1]$.

- $\mathbf{r}(t_0) = (x(t_0), y(t_0)) = (1 - t_0^2, t_0) = (1, 0) \Rightarrow t_0 = 0$.
- Velocity $\mathbf{v}(t_0) = (x'(t_0), y'(t_0)) = (-2t_0, 1) = (0, 1)$.
- Speed $v(t_0) = \|\mathbf{v}(t_0)\| = \sqrt{[x'(t_0)]^2 + [y'(t_0)]^2} = 1$.
- Unit tangent vector $\mathbf{T}(t_0) = \frac{\mathbf{v}(t_0)}{v(t_0)} = (0, 1)$
- Tangent line $x = 1$

Example: Parabola Arc $x = 1 - y^2$, $-1 \le y \le 1$



Point of the Vertical Tangent: $\mathbf{r}(t_0) = (1, 0)$

Curve: $x(t) = 1 - t^2$, $y(t) = t$, $t \in [-1, 1]$.

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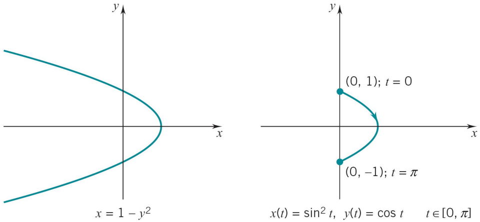
Example: Parabola Arc $x = 1 - y^2$, $-1 \le y \le 1$



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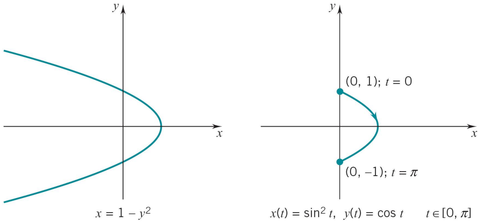

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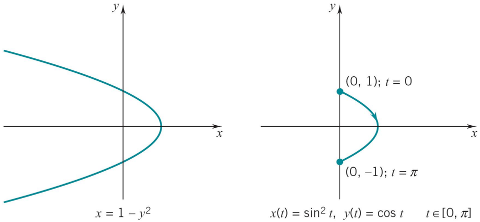

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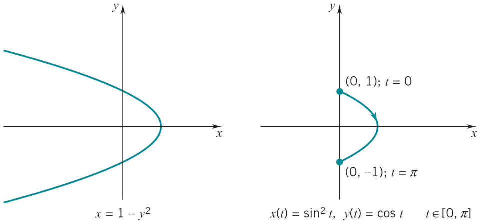

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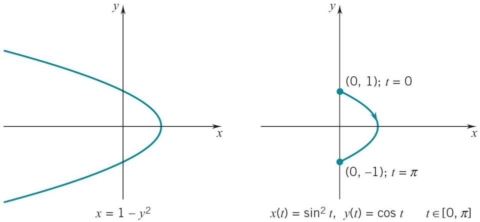

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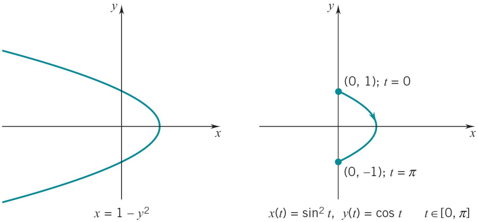

Point of the Vertical Tangent: $\mathbf{r}(t_0) = (1, 0)$

Curve: $x(t) = 1 - \cos^2 t$, $y(t) = \cos t$, $t \in [0, \pi]$

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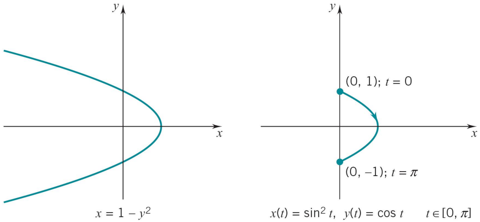

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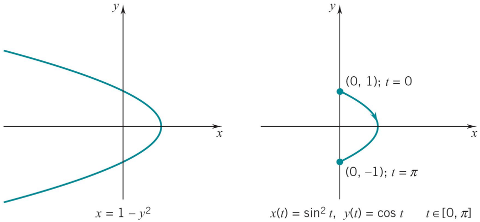

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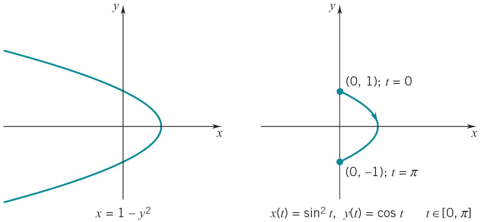

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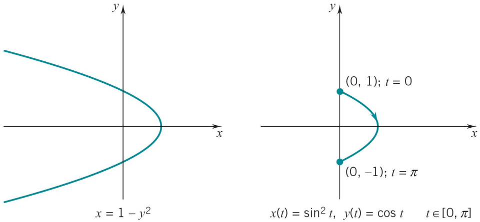

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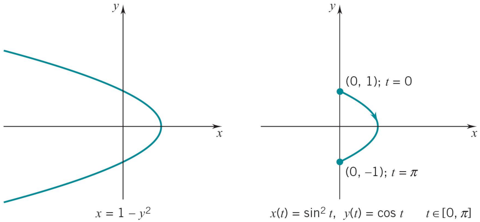

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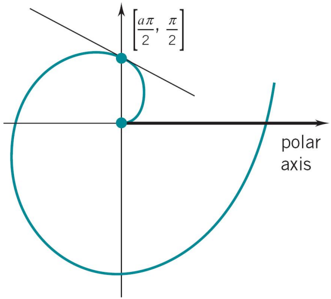
Example: Spiral of Archimedes



Slope of the Spiral of Archimedes $r = \theta$ at $\theta_0 = \frac{\pi}{2}$

- $\mathbf{r}(\theta_0) = \big(x(\theta_0), y(\theta_0)\big) = \big(\theta_0 \cos\theta_0, \theta_0 \sin\theta_0\big) = (0, \frac{\pi}{2})$.
- $\mathbf{v}(\theta_0) = \mathbf{r}'(\theta_0) = \big(\cos\theta_0 - \theta_0 \sin\theta_0, \sin\theta_0 + \theta_0\cos\theta_0\big) = (-\frac{\pi}{2}, 1)$.
- Slope $m(\theta_0) = \frac{y'(\theta_0)}{x'(\theta_0)} = -\frac{2}{\pi}$.
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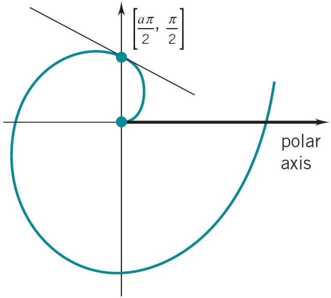

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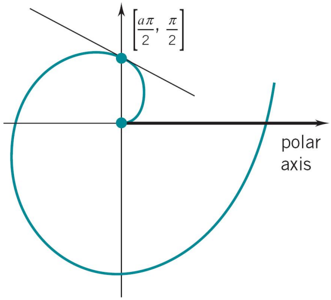

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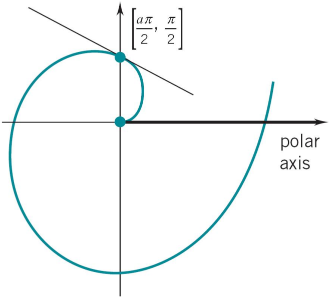

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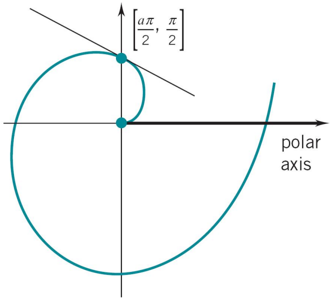

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Quiz

Quiz

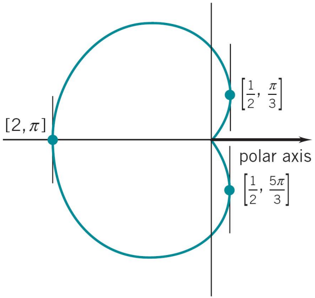
1. $r = 3 + 3\cos\theta$ is a

   (a) cardioid, (b) circle, (c) limacon with an inner loop.

2. $r = 2\sin\theta$ is a

   (a) cardioid, (b) circle, (c) limacon with an inner loop.

Example: Limaçon



Point of Vertical Tangent for Limaçon (Snail): $r = 1 - \cos\theta$

- $\mathbf{r}(t) = (x(t), y(t)) = ((1-\cos t)\cos t, (1-\cos t)\sin t)$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = ((2\cos t - 1)\sin t, (1-\cos t)(1+2\cos t))$.
- Set $x'(t) = 0$, $\cos t = \frac{1}{2}$ or $\sin t = 0$, then $t = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.
- Tangent line is vertical at $\mathbf{r}(t) = (\frac{1}{4}, \frac{\sqrt{3}}{4}), (-2, 0), (\frac{1}{4}, -\frac{\sqrt{3}}{4})$.

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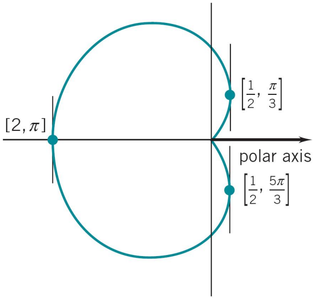

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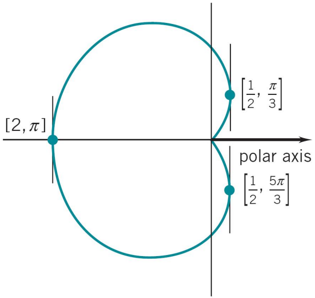

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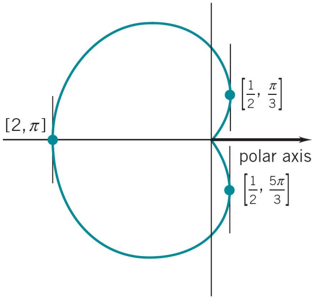

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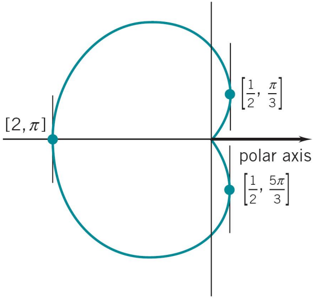
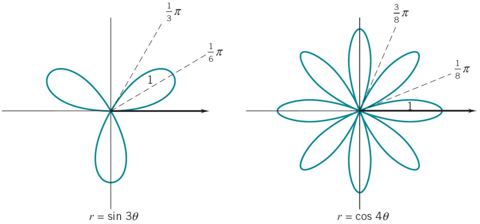


Point of Vertical Tangent for Limaçon (Snail): $r = 1 - \cos\theta$

- $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big((1-\cos t)\cos t, (1-\cos t)\sin t\big)$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = \big((2\cos t - 1)\sin t, (1-\cos t)(1+2\cos t)\big)$.
- Set $x'(t) = 0$, $\cos t = \frac{1}{2}$ or $\sin t = 0$, then $t = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.
- Tangent line is vertical at $\mathbf{r}(t) = (\frac{1}{4}, \frac{\sqrt{3}}{4}), (-2, 0), (\frac{1}{4}, -\frac{\sqrt{3}}{4})$.

Example: Petal Curves (Flowers)



Tangent Lines at the Origin: $r = \sin 3\theta$

- The curve passes through the origin when $r = \sin 3\theta = 0$, i.e, at $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$.
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- Slope $m(t) = \frac{y'(t)}{x'(t)} = 0, \sqrt{3}, -\sqrt{3}$.
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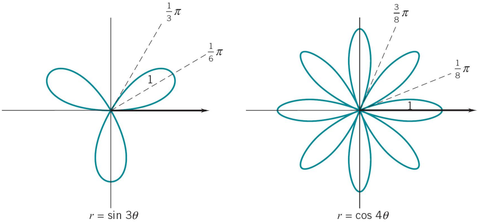
Example: Petal Curves (Flowers)



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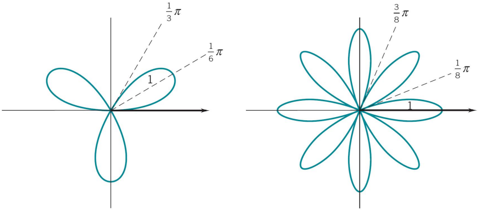

$r = \sin 3\theta$ $\quad$ $r = \cos 4\theta$

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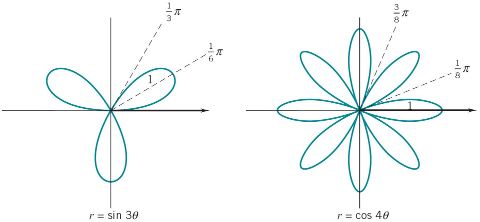

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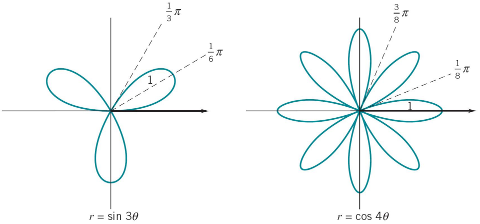
Example: Petal Curves (Flowers)



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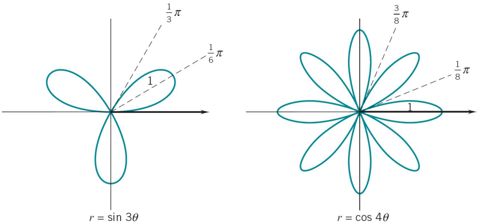

Tangent Lines at the Origin: $r = \cos 4\theta$

- The curve passes through the origin when $r = \cos 4\theta = 0$, i.e, at $\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$.
- $\mathbf{r}(t) = (x(t), y(t)) = (\cos 4 \cos t, \cos 4t \sin t) = (0, 0)$.
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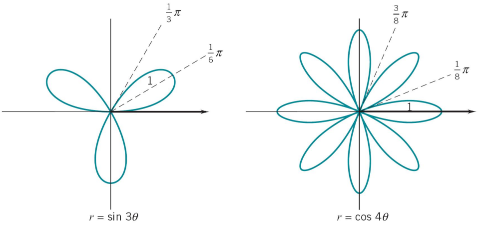

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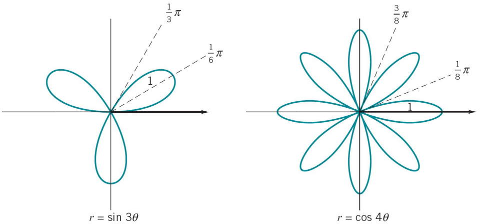

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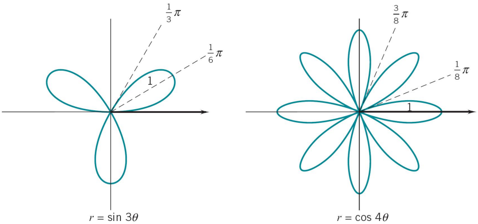

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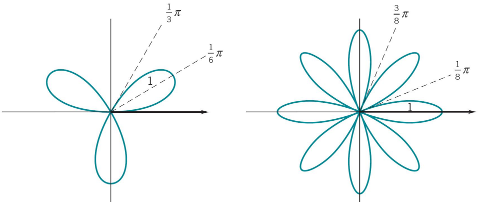

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Quiz

Quiz

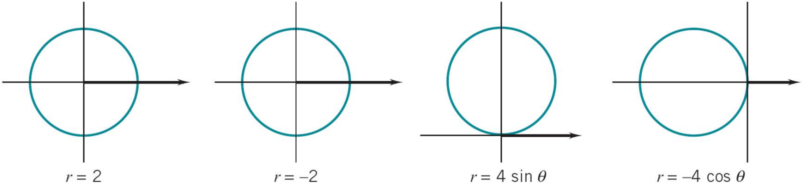
3. $r = 2\sin 3\theta$ is a

   (a) flower with 6 petals, (b) circle, (c) flower with 3 petals.

4. The curve $x(t) = 3\cos t,\ y(t) = 2\sin t,\ t \in [0, 2\pi]$ is:

   (a) circle, (b) parabola, (c) ellipse.

Circles: $C = \{P : d(P, O) = |a|\}$



Horizontal and Vertical Tangent for Circle Centered at (c, d)

- $\mathbf{r}(t) = (x(t), y(t)) = (a \cos t + c, a \sin t + d), t \in [0, 2\pi)$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = (-a \sin t, a \cos t))$.
- Set $x'(t) = 0$, $\sin t = 0$, then $t = 0, \pi$;
  set $y'(t) = 0$, $\cos t = 0$, then $t = \frac{\pi}{2}, \frac{3\pi}{2}$
- Tangent line is horizontalat $\mathbf{r}(t) = (c, d + a), (c, d - a)$;
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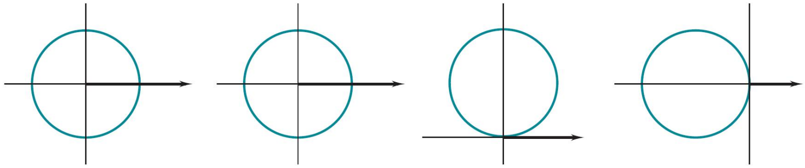

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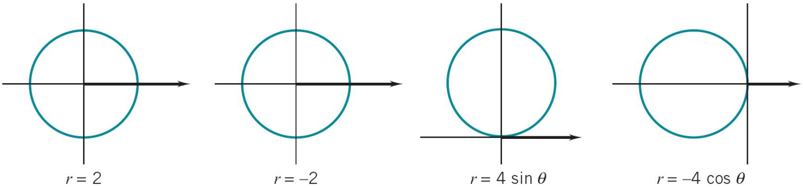

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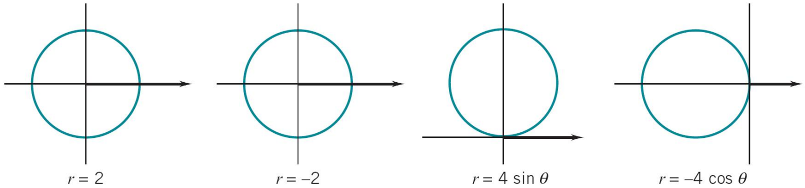

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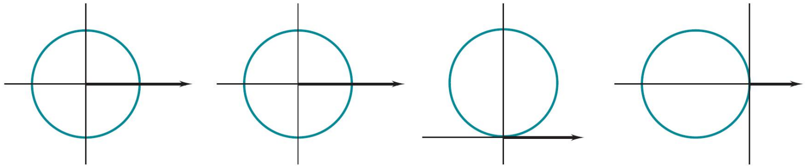

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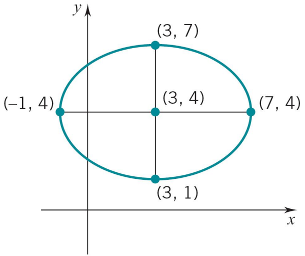
Ellipses: Cosine and Sine



Horizontal and Vertical Tangent for Ellipse Centered at (d, e)

- $\mathbf{r}(t) = (x(t), y(t)) = (a\cos t + d, b\sin t + e), t \in [0, 2\pi)$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = (-a\sin t, b\cos t))$.
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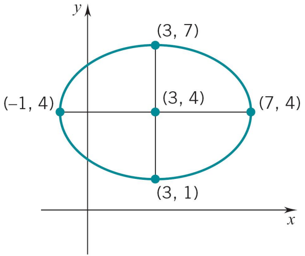

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- $\mathbf{v}(t) = \mathbf{r}'(t) = \big(-a\sin t, b\cos t\big)$.
- Set $x'(t) = 0$, $\sin t = 0$, then $t = 0, \pi$;
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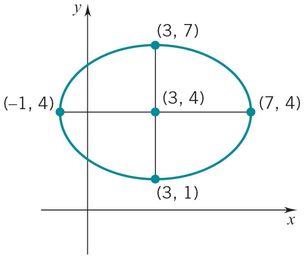

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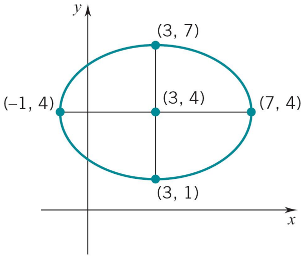
Ellipses: Cosine and Sine



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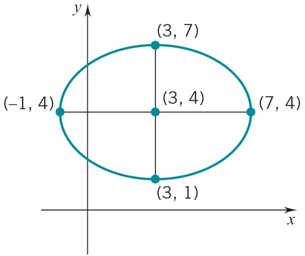
Ellipses: Cosine and Sine



Horizontal and Vertical Tangent for Ellipse Centered at (d, e)

- $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big(a\cos t + d, b\sin t + e\big), t \in [0, 2\pi)$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = \big(-a\sin t, b\cos t\big)$.
- Set $x'(t) = 0$, $\sin t = 0$, then $t = 0, \pi$;
  set $y'(t) = 0$, $\cos t = 0$, then $t = \frac{\pi}{2}, \frac{3\pi}{2}$
- Tangent line is horizontalat $\mathbf{r}(t) = (d, e + a), (d, e - a)$;
  it is vertical at $\mathbf{r}(t) = (d + b, e), (d - a, e)$.

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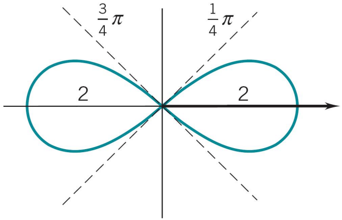
Lemniscates (Ribbons): $r^2 = a^2 \cos 2\theta$



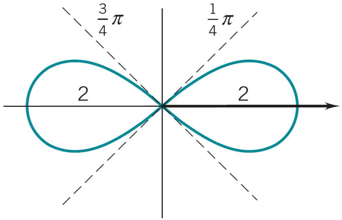
Tangent Lines at the Origin

- $\mathbf{r}(t) = (x(t), y(t)) = (0,0) \quad \Rightarrow \quad t = \frac{\pi}{2}, \frac{3\pi}{2}$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = \left(-\frac{a}{2}, -\frac{a}{2}\right), \left(\frac{a}{2}, -\frac{a}{2}\right)$.
- Slope $m(t) = \frac{y'(t)}{x'(t)} = 1, -1$. $\Rightarrow \quad \theta_0 = \tan^{-1}(1) = \frac{\pi}{4}$, $\theta_1 = \tan^{-1}(-1) = \frac{3\pi}{4}$
- Tangent line at the origin $y = x$, $y = -x$.

The parametric equations for the lemniscate with $a^2 = 2c^2$ is

$$x = \frac{a \cos t}{1 + \sin^2 t}, \quad y = \frac{a \sin t \cos t}{1 + \sin^2 t}, \quad t \in (0, 2\pi).$$

Lemniscates (Ribbons): $r^2 = a^2 \cos 2\theta$



$r^2 = 4 \cos 2\theta$

Tangent Lines at the Origin

- $\mathbf{r}(t) = (x(t), y(t)) = (0, 0) \quad \Rightarrow$ $t = \frac{\pi}{2}, \frac{3\pi}{2}$.
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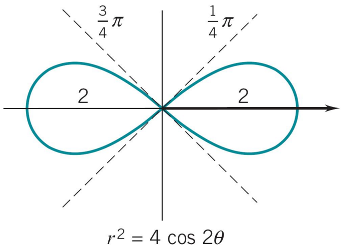
Lemniscates (Ribbons): $r^2 = a^2 \cos 2\theta$



Tangent Lines at the Origin

- $\mathbf{r}(t) = \big(x(t), y(t)\big) = (0,0) \quad \Rightarrow$ $t = \frac{\pi}{2}, \frac{3\pi}{2}$.
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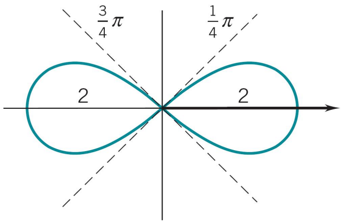
Lemniscates (Ribbons): $r^2 = a^2 \cos 2\theta$



$r^2 = 4\cos 2\theta$

Tangent Lines at the Origin

- $\mathbf{r}(t) = \big(x(t), y(t)\big) = (0,0) \quad \Rightarrow$ $t = \frac{\pi}{2}, \frac{3\pi}{2}$.
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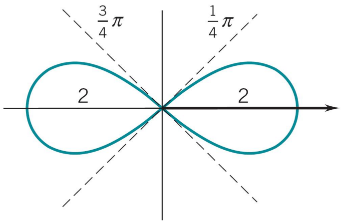
Lemniscates (Ribbons): $r^2 = a^2 \cos 2\theta$



Tangent Lines at the Origin

- $\mathbf{r}(t) = \big(x(t), y(t)\big) = (0,0) \quad \Rightarrow$
  $t = \frac{\pi}{2}, \frac{3\pi}{2}$.
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Outline

- Tangents to Parametrized curves
  - Tangents to Parametrized curve
  - Examples
- Locus
  - Circles
  - Ellipses