Lecture 15 Section 9.7 Tangents to Curves Given Parametrically

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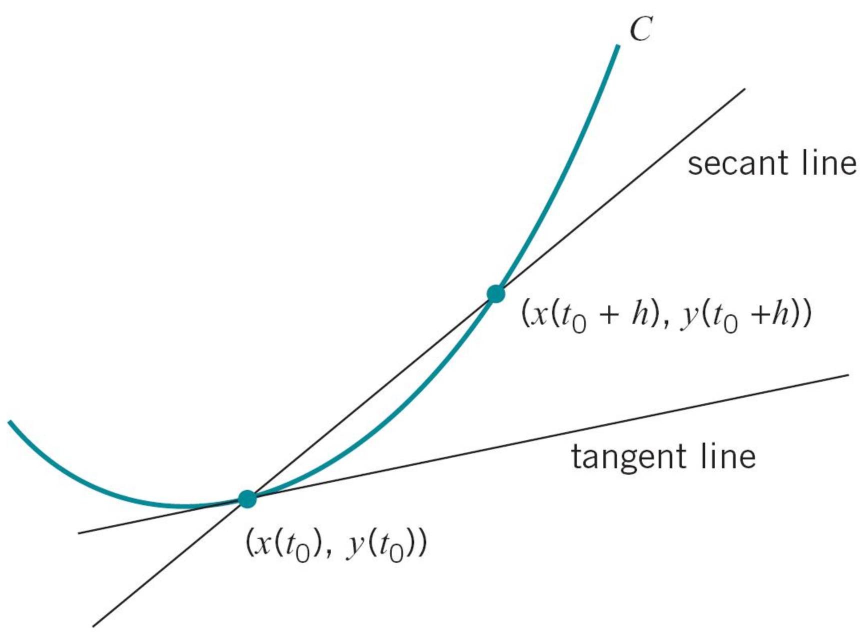
1 Tangents to Parametrized curves

1.1 Tangents to Parametrized curve

Tangents to Parametrized curves



Tangent line

Let $C = \{(x(t), y(t)) : t \in I\}$. For a time $t_0 \in I$, assume $x'(t_0) \neq 0$. The *slope of the curve* at *time* t_0 is

$$m(t_0) = \frac{y'(t_0)}{x'(t_0)}$$

The equation of the *tangent line* is

$$x'(t_0)\big(y - y_0\big) - y'(t_0)\big(x - x_0\big) = 0$$

Proof.

$$m(t_0) = \lim_{h\to 0} \frac{y(t_0+h) - y(t_0)}{x(t_0+h) - x(t_0)} = \frac{\displaystyle\lim_{h\to 0} \frac{y(t_0+h) - y(t_0)}{h}}{\displaystyle\lim_{h\to 0} \frac{x(t_0+h) - x(t_0)}{h}} = \frac{y'(t_0)}{x'(t_0)}$$

$$\big(y - y_0\big) = \frac{y'(t_0)}{x'(t_0)}\big(x - x_0\big) \quad \Rightarrow \quad x'(t_0)\big(y - y_0\big) = y'(t_0)\big(x - x_0\big)$$

Definition

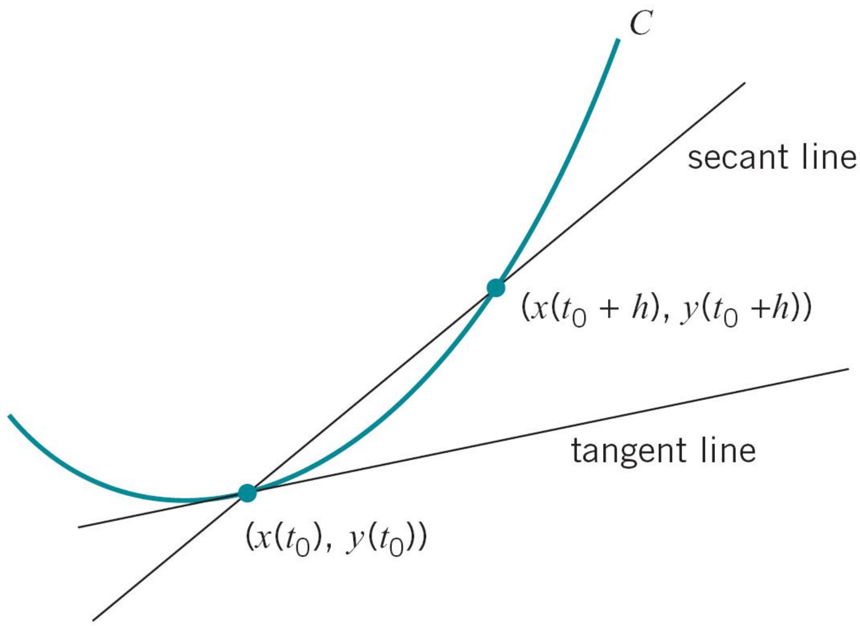
- The curve has a *vertical tangent* if $x'(t_0) = 0$
- The curve has a *horizontal tangent* if $y'(t_0) = 0$

Example 1. • The graph of a *function* $y = f(x)$, $x \in I$, is a curve C that is *parametrized* by $x(t) = t$, $y(t) = f(t)$, $t \in I$.

- The *slope of the curve* at *time* t_0 is

$$m(t_0) = \frac{y'(t_0)}{x'(t_0)} = \frac{dy}{dx}(x_0) = f'(x_0), \quad x_0 = t_0$$

Velocity and Speed Along a Plane Curve



Parametrization by the Motion

- Imaging an object *moving along the curve* C.
- Let $\mathbf{r}(t) = \big(x(t), y(t)\big)$ the *position* of the object at time t.

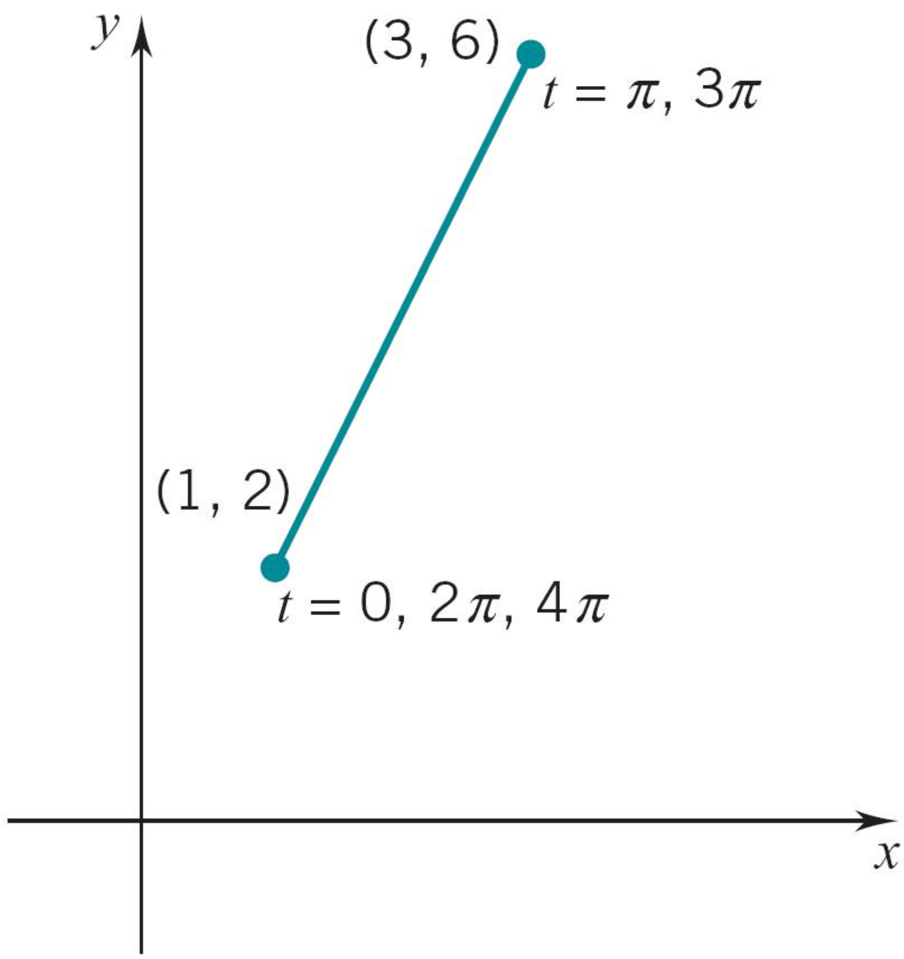
Velocity and Speed Along a Plane Curve

- The *velocity* of the object at time t is $\mathbf{v}(t) = \mathbf{r}'(t) = \big(x'(t), y'(t)\big)$.
- The *speed* of the object at time t is $v(t) = \|\mathbf{v}(t)\| = \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2}$.
- The *instantaneous direction of motion* gives the *unit tangent vector* $\mathbf{T}$:

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)}.$$

1.2 Examples

Example: Line Segment



Line Segment: $y = 2x,\ x \in [1, 3]$

- Set $x(t) = t$, then $y(t) = 2t$, $t \in [1, 3]$
- Set $x(t) = 3 - t$, then $y(t) = 6 - 2t$, $t \in [0, 2]$
- Set $x(t) = 2 - \cos t$, then $y(t) = 4 - 2\cos t$, $t \in [0, 4\pi]$.

At time $t \in [1, 3]$:

- The *position* $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big(t, 2t\big)$.
- The *velocity* $\mathbf{v}(t) = \big(x'(t), y'(t)\big) = \big(1, 2\big)$.
- The *speed* $v(t) = \|\mathbf{v}(t)\| = \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2} = \sqrt{5}$.
- The *unit tangent vector* $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{1}{\sqrt{5}}\big(1, 2\big)$
- The *slope* $m(t) = \frac{y'(t)}{x'(t)} = 2$.
- The *tangent line* $y = 2x$

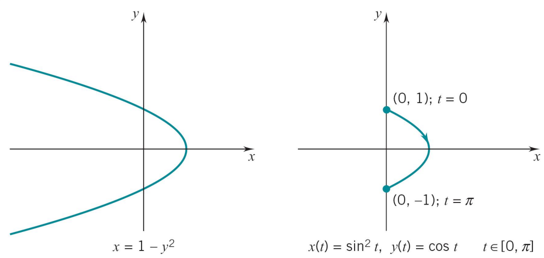
At time $t \in [0, 2]$:

- The *position* $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big(3 - t, 6 - 2t\big)$.
- The *velocity* $\mathbf{v}(t) = \big(x'(t), y'(t)\big) = \big(-1, -2\big)$.
- The *speed* $v(t) = \|\mathbf{v}(t)\| = \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2} = \sqrt{5}$.
- The *unit tangent vector* $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{1}{\sqrt{5}}\big(-1, -2\big)$
- The *slope* $m(t) = \frac{y'(t)}{x'(t)} = 2$.
- The *tangent line* $y = 2x$

At time $t \in [0, 4\pi]$:

- The *position* $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big(2 - \cos t, 4 - 2\cos t\big)$.
- The *velocity* $\mathbf{v}(t) = \big(x'(t), y'(t)\big) = \big(\sin t, 2\sin t\big)$.
- The *speed* $v(t) = \|\mathbf{v}(t)\| = \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2} = \sqrt{5}\sin t$.
- The *unit tangent vector* $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{1}{\sqrt{5}}\big(1, 2\big)$
- The *slope* $m(t) = \frac{y'(t)}{x'(t)} = 2$.
- The *tangent line* $y = 2x$

Example: Parabola Arc $x = 1 - y^2$, $-1 \le y \le 1$



Point of the Vertical Tangent: $\mathbf{r}(t_0) = (1, 0)$

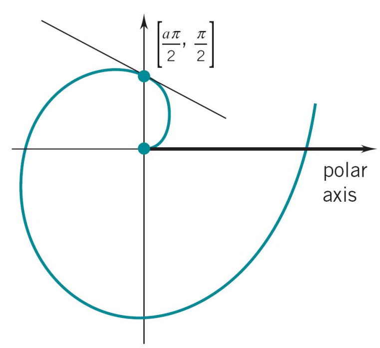
Curve: $x(t) = 1 - t^2$, $y(t) = t$, $t \in [-1, 1]$.

- $\mathbf{r}(t_0) = \big(x(t_0), y(t_0)\big) = \big(1 - t_0^2, t_0\big) = (1, 0) \Rightarrow t_0 = 0.$
- *Velocity* $\mathbf{v}(t_0) = \big(x'(t_0), y'(t_0)\big) = \big(-2t_0, 1\big) = (0, 1).$
- *Speed* $v(t_0) = \|\mathbf{v}(t_0)\| = \sqrt{\big[x'(t_0)\big]^2 + \big[y'(t_0)\big]^2} = 1.$
- *Unit tangent vector* $\mathbf{T}(t_0) = \frac{\mathbf{v}(t_0)}{v(t_0)} = \big(0, 1\big)$
- *Tangent line* $x = 1$

Curve: $x(t) = 1 - \cos^2 t$, $y(t) = \cos t$, $t \in [0, \pi]$

- $\mathbf{r}(t_0) = \big(1 - \cos^2 t_0, \cos t_0\big) = (1, 0) \Rightarrow t_0 = \frac{\pi}{2}.$
- $\mathbf{v}(t_0) = \big(x'(t_0), y'(t_0)\big) = \big(\sin 2t_0, -\sin t_0\big) = (0, -1).$
- *Speed* $v(t_0) = \|\mathbf{v}(t_0)\| = \sqrt{\big[x'(t_0)\big]^2 + \big[y'(t_0)\big]^2} = 1.$
- *Unit tangent vector* $\mathbf{T}(t_0) = \frac{\mathbf{v}(t_0)}{v(t_0)} = \big(0, -1\big)$
- *Tangent line* $x = 1$

Example: Spiral of Archimedes



Slope of the Spiral of Archimedes $r = \theta$ at $\theta_0 = \frac{\pi}{2}$

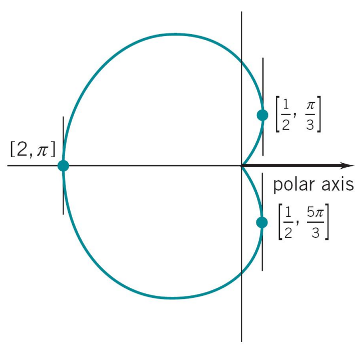
- $\mathbf{r}(\theta_0) = \big(x(\theta_0), y(\theta_0)\big) = \big(\theta_0 \cos\theta_0, \theta_0 \sin\theta_0\big) = (0, \frac{\pi}{2})$.
- $\mathbf{v}(\theta_0) = \mathbf{r}'(\theta_0) = \big(\cos\theta_0 - \theta_0 \sin\theta_0, \sin\theta_0 + \theta_0 \cos\theta_0\big) = (-\frac{\pi}{2}, 1)$.
- *Slope* $m(\theta_0) = \frac{y'(\theta_0)}{x'(\theta_0)} = -\frac{2}{\pi}$.
- *Tangent line* at θ_0 $y = \frac{\pi}{2} - \frac{2}{\pi}x$.

Quiz

Quiz

1. $r = 3 + 3\cos\theta$ is a
   (a) cardioid, (b) circle, (c) limacon with an inner loop.
2. $r = 2\sin\theta$ is a
   (a) cardioid, (b) circle, (c) limacon with an inner loop.

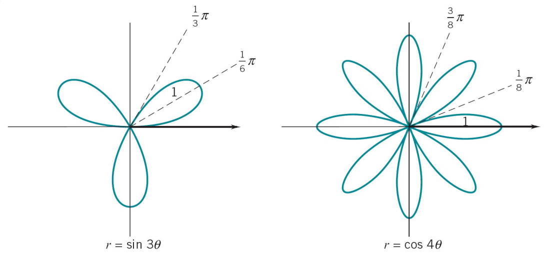
Example: Limaçon



Point of Vertical Tangent for Limaçon (Snail): $r = 1 - \cos\theta$

- $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big((1-\cos t)\cos t, (1-\cos t)\sin t\big)$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = \big((2\cos t - 1)\sin t, (1-\cos t)(1+2\cos t)\big)$.
- *Set* $x'(t) = 0$, $\cos t = \frac{1}{2}$ or $\sin t = 0$, then $t = \frac{\pi}{3}$, π, $\frac{5\pi}{3}$.
- *Tangent line* is vertical at $\mathbf{r}(t) = (\frac{1}{4}, \frac{\sqrt{3}}{4})$, $(-2, 0)$, $(\frac{1}{4}, -\frac{\sqrt{3}}{4})$.

Example: Petal Curves (Flowers)



Tangent Lines at the Origin: $r = \sin 3\theta$

- The curve passes through the origin when $r = \sin 3\theta = 0$, i.e, at $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$.
- $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big(\sin 3t \cos t, \sin 3t \sin t\big) = (0, 0)$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = \big(3\cos 3t \cos t - \sin 3t \sin t, 3\cos 3t \sin t + \sin 3t \cos t\big) = \big(3, 0\big), \big(-\frac{3}{2}, -\frac{3\sqrt{3}}{2}\big), \big(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\big)$.
- *Slope* $m(t) = \frac{y'(t)}{x'(t)} = 0, \sqrt{3}, -\sqrt{3}$.
- *Tangent line* at the origin $y = 0$, $y = \sqrt{3}x$, $y = -\sqrt{3}x$.

Tangent Lines at the Origin: $r = \cos 4\theta$

- The curve passes through the origin when $r = \cos 4\theta = 0$, i.e, at $\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$.
- $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big(\cos 4 \cos t, \cos 4t \sin t\big) = (0, 0)$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = \big(-4\sin 4t \cos t - \cos 4t \sin t, -4\sin 4t \sin t + \cos 4t \cos t\big) = \big(-4\cos\frac{\pi}{8}, -4\sin\frac{\pi}{8}\big), \big(4\cos\frac{3\pi}{8}, 4\sin\frac{3\pi}{8}\big), \cdots$.
- *Slope* $m(t) = \frac{y'(t)}{x'(t)} = \tan\frac{\pi}{8}, \tan\frac{3\pi}{8}, \cdots$.
- *Tangent line* at the origin $y = \tan\frac{\pi}{8}x$, $y = \tan\frac{3\pi}{8}x$, $\cdots$.

Quiz

Quiz

3. $r = 2\sin 3\theta$ is a

   (a) flower with 6 petals, (b) circle, (c) flower with 3 petals.

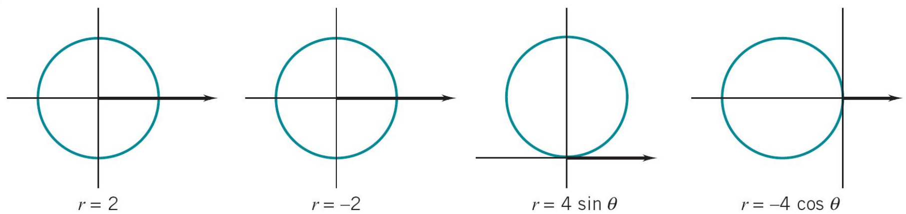
4. The curve $x(t) = 3\cos t,\ y(t) = 2\sin t,\ t \in [0, 2\pi]$ is:

   (a) circle, (b) parabola, (c) ellipse.

2 Locus

2.1 Circles

Circles: $\mathbf{C} = \{P : d(P, O) = |a|\}$

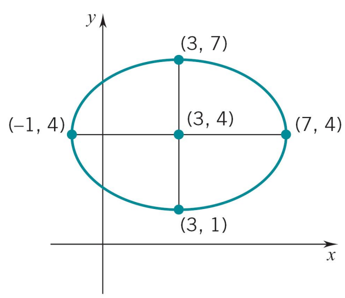


Horizontal and Vertical Tangent for Circle Centered at (c, d)

- $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big(a\cos t + c, a\sin t + d\big), t \in [0, 2\pi)$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = \big(-a\sin t, a\cos t\big)$.
- *Set* $x'(t) = 0$, $\sin t = 0$, then $t = 0, \pi$; *set* $y'(t) = 0$, $\cos t = 0$, then $t = \frac{\pi}{2}, \frac{3\pi}{2}$
- *Tangent line* is *horizontal*at $\mathbf{r}(t) = (c, d + a), (c, d - a)$; it is *vertical* at $\mathbf{r}(t) = (c + a, d), (c - a, d)$.

2.2 Ellipses

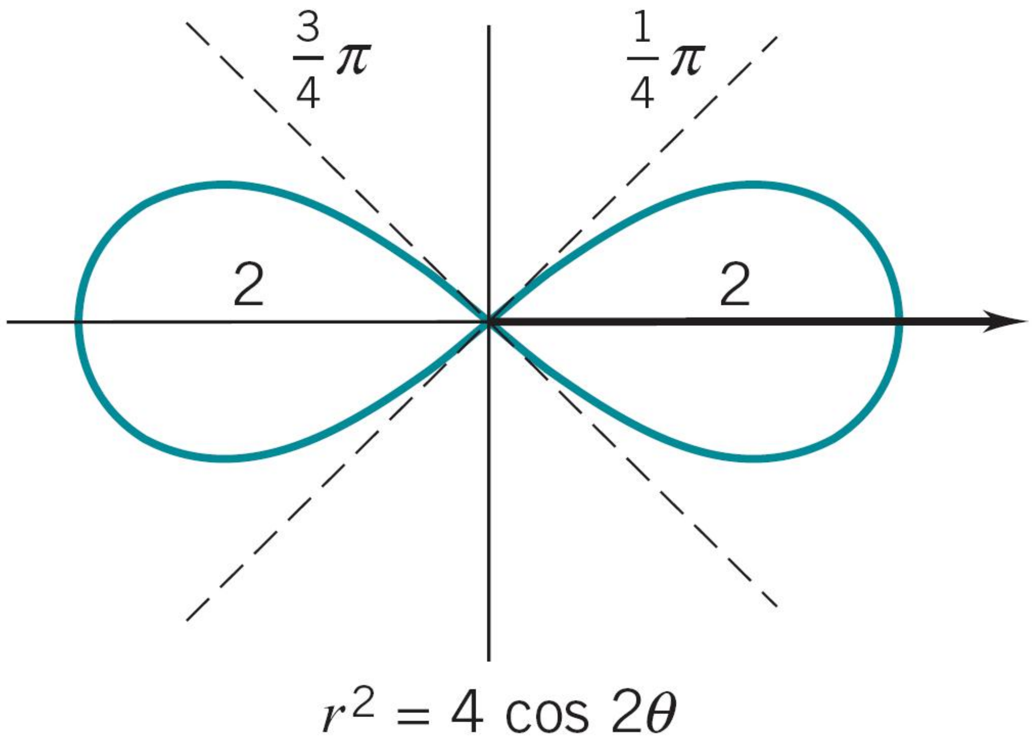
Ellipses: Cosine and Sine



Horizontal and Vertical Tangent for Ellipse Centered at (d, e)

- $\mathbf{r}(t) = \big(x(t), y(t)\big) = \big(a\cos t + d, b\sin t + e\big), t \in [0, 2\pi)$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = \big(-a\sin t, b\cos t\big)$.
- *Set* $x'(t) = 0$, $\sin t = 0$, then $t = 0, \pi$; *set* $y'(t) = 0$, $\cos t = 0$, then $t = \frac{\pi}{2}, \frac{3\pi}{2}$
- *Tangent line* is *horizontal*at $\mathbf{r}(t) = (d, e + a), (d, e - a)$; it is *vertical* at $\mathbf{r}(t) = (d + b, e), (d - a, e)$.

Lemniscates (Ribbons): $r^2 = a^2 \cos 2\theta$



Tangent Lines at the Origin

- $\mathbf{r}(t) = \big(x(t), y(t)\big) = (0,0) \quad \Rightarrow \quad t = \frac{\pi}{2}, \frac{3\pi}{2}$.
- $\mathbf{v}(t) = \mathbf{r}'(t) = \big(-\frac{a}{2}, -\frac{a}{2}\big), \big(\frac{a}{2}, -\frac{a}{2}\big)$.
- *Slope* $m(t) = \frac{y'(t)}{x'(t)} = 1, -1. \quad \Rightarrow \quad \theta_0 = \tan^{-1}(1) = \frac{\pi}{4},\ \theta_1 = \tan^{-1}(-1) = \frac{3\pi}{4}$
- *Tangent line* at the origin $y = x$, $y = -x$.

The *parametric equations* for the lemniscate with $a^2 = 2c^2$ is

$$x = \frac{a\,\cos t}{1+\sin^2 t}, \quad y = \frac{a\,\sin t \cos t}{1+\sin^2 t}, \quad t \in (0, 2\pi).$$

Outline

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