Lecture 16

Section 9.8 Arc Length and Speed

Jiwen He

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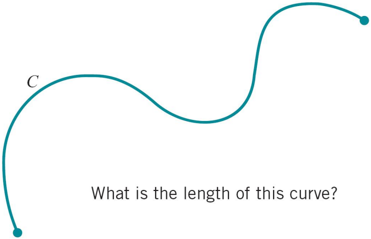
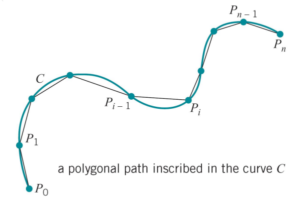
http://math.uh.edu/∼jiwenhe/Math1432



What is the length of this curve?

Arc Length Formulas



Arc Length Formulas

Let $C = \{(x(t), y(t)) : t \in I\}$.

The length of C is

$$L(C) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$\begin{aligned} d(P_{i-1}, P_i) &= \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2} \\ &= \sqrt{\left[\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}\right]^2 + \left[\frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}}\right]^2} (t_i - t_{i-1}) \\ &= \sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2}\, \Delta t_i \end{aligned}$$

$$\begin{aligned} L(\gamma) &= d(P_0, P_1) + \cdots + d(P_{i-1}, P_i) + \cdots + d(P_{n-1}, P_n) \\ &= \sum_{i=1}^{n} \sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2}\, \Delta t_i \to L(C) \quad \text{as } \Delta t_i \to 0. \end{aligned}$$

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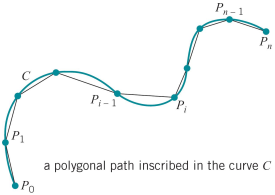

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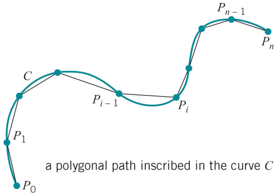

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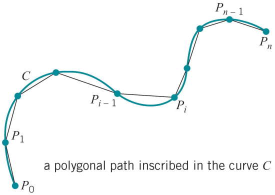

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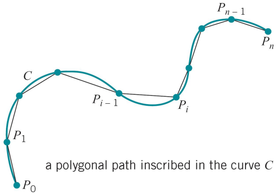

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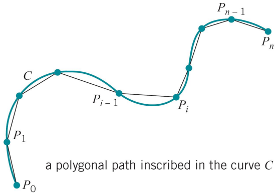

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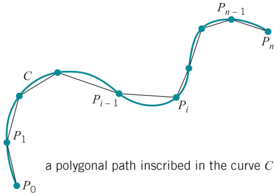

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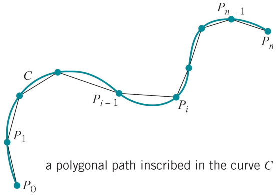

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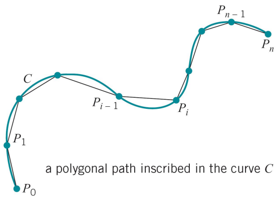

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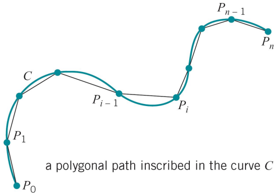

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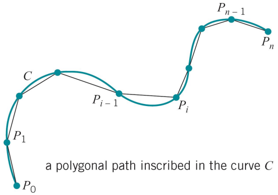

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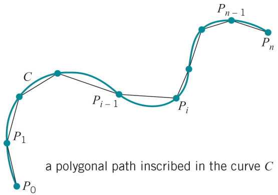

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- We define the element of length ds

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

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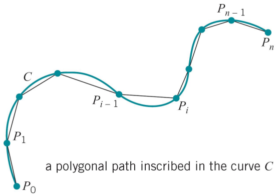

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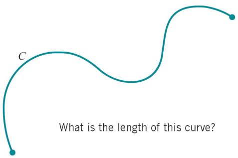

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- Imaging an object moving along the curve C.
- Let $\mathbf{r}(t) = (x(t), y(t))$ the position of the object at time t.
- The velocity of the object at time t is $\mathbf{v}(t) = \mathbf{r}'(t) = (x'(t), y'(t))$.

Arc Length and Speed Along a Plane Curve

- The speed of the object at time t is
$$v(t) = \|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}.$$
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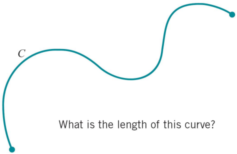

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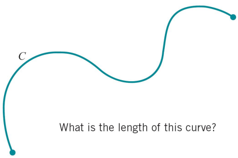

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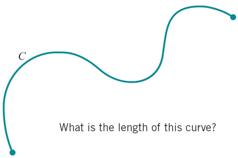

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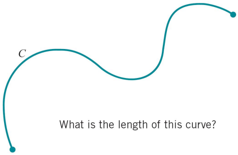

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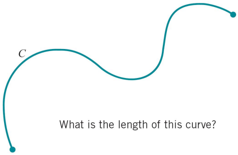

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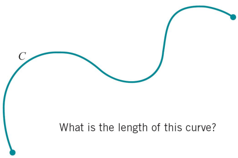


Parametrization by the Motion

- Imaging an object moving along the curve C.
- Let $\mathbf{r}(t) = \big(x(t), y(t)\big)$ the position of the object at time t.
- The velocity of the object at time t is $\mathbf{v}(t) = \mathbf{r}'(t) = \big(x'(t), y'(t)\big)$.

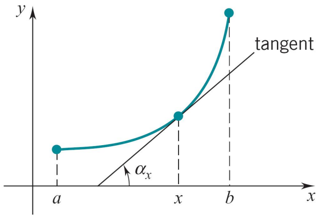
Arc Length and Speed Along a Plane Curve

- The speed of the object at time t is $v(t) = \|\mathbf{v}(t)\| = \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2}$.
- The distance traveled by the object from time zero to any later time t is

$$s(t) = \int ds = \int_0^t \sqrt{\big[x'(u)\big]^2 + \big[y'(u)\big]^2}\, du = \int_0^t v(u)\, du.$$

- We have $ds = v(t)\, dt$.

Length of the Arc on the Graph of $y = f(x)$



Length of $y = f(x)$, $x \in [a, b]$

The length of the arc on the graph from a to x is

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

$$\Rightarrow \quad ds = \sqrt{1 + [f'(x)]^2}\, dx.$$

Proof.

Set $x(t) = t$, $y(t) = f(t)$, $t \in [a, b]$.

Since $x'(t) = 1$, $y'(t) = f'(t)$, then

$$s(x) = \int_a^x \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

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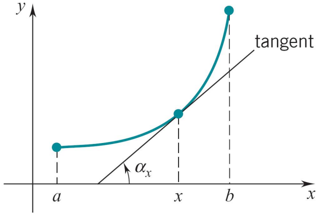

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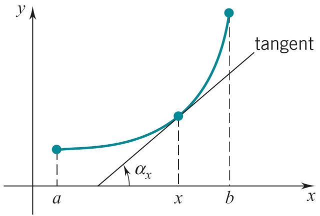

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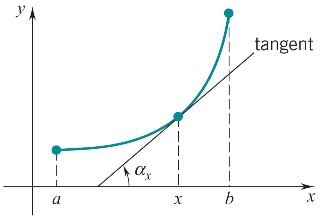

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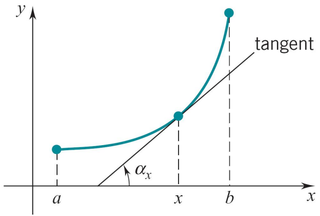

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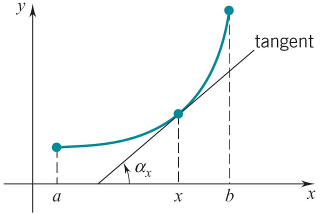

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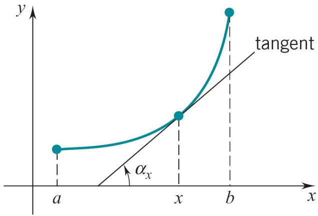

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$$= \left[x\sqrt{\frac{1}{4} + x^2} + \frac{1}{4}\ln\left(x + \sqrt{\frac{1}{4} + x^2}\right)\right]_0^1 = \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(2 + \sqrt{5}).$$

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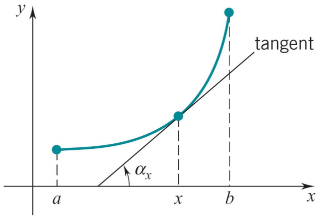

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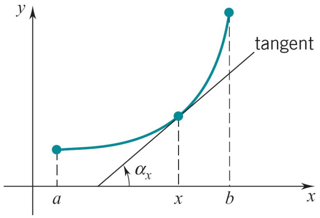

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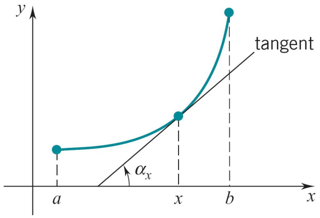

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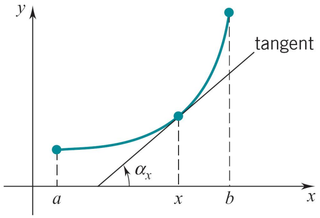

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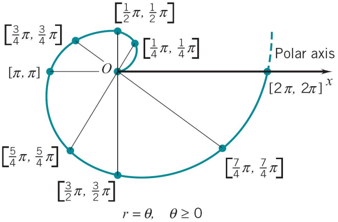
Length of the Arc on the Graph of $r = \rho(\theta)$



spiral of Archimedes

Length of $r = \rho(\theta)$, $\theta \in [\alpha, \beta]$

The length of the arc on the graph from α to θ is

$$s(\theta) = \int_\alpha^\theta \sqrt{[\rho(t)]^2 + [\rho'(t)]^2}\, dt.$$

$$\Rightarrow \quad ds(\theta) = \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2}\, d\theta$$

Proof.

Set $x(t) = \rho(t)\cos t$, $y(t) = \rho(t)\sin t$, $t \in [\alpha, \beta]$.

Since $[x'(t)]^2 + [y'(t)]^2 = [\rho(t)]^2 + [\rho'(t)]^2$, then

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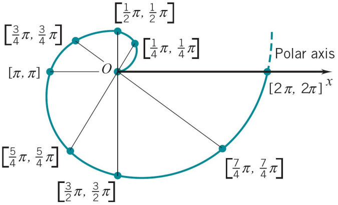

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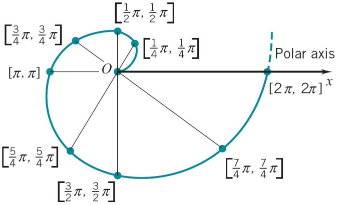

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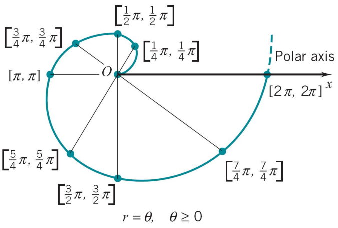

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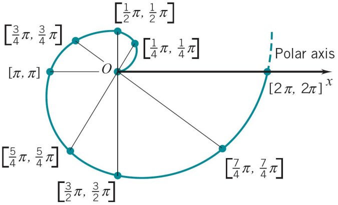

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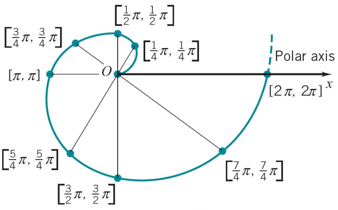

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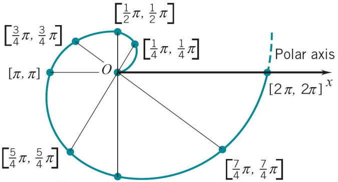

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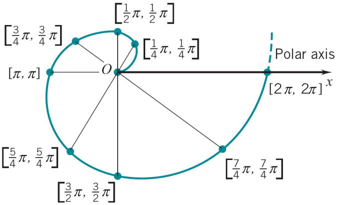

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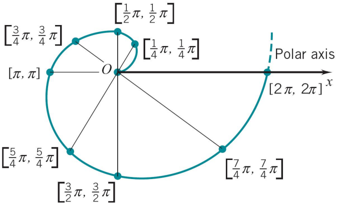

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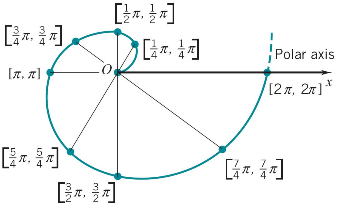

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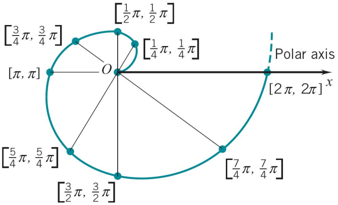


$r = \theta, \quad \theta \geq 0$

spiral of Archimedes

Length of $r = \rho(\theta)$, $\theta \in [\alpha, \beta]$

The length of the arc on the graph from α to θ is

$$s(\theta) = \int_\alpha^\theta \sqrt{\big[\rho(t)\big]^2 + \big[\rho'(t)\big]^2}\, dt.$$

$$\Rightarrow \quad ds(\theta) = \sqrt{\big[\rho(\theta)\big]^2 + \big[\rho'(\theta)\big]^2}\, d\theta$$

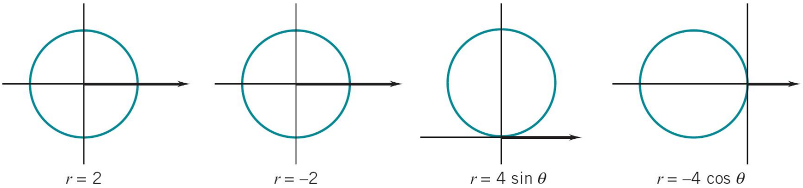
Spiral of Archimedes: $r = \theta$, $\theta \geq 0$

- The length of the arc: $r = \theta$, $\theta \in [0, 2\pi]$, is given

$$\int_0^{2\pi} \sqrt{\big[\rho(\theta)\big]^2 + \big[\rho'(\theta)\big]^2}\, d\theta = \int_0^{2\pi} \sqrt{1+\theta^2}\, d\theta$$

$$= \left[\frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln(\theta + \sqrt{1+\theta^2})\right]_0^{2\pi} = \cdots$$

Example: Circle of Radius a: $L = 2\pi a$



Circle in Polar Coordinates

$$r = a, \quad 0 \le \theta \le 2\pi$$

$$L = \int_0^{2\pi} \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2}\, d\theta = \int_0^{2\pi} \sqrt{a^2 + 0}\, d\theta = 2\pi a$$

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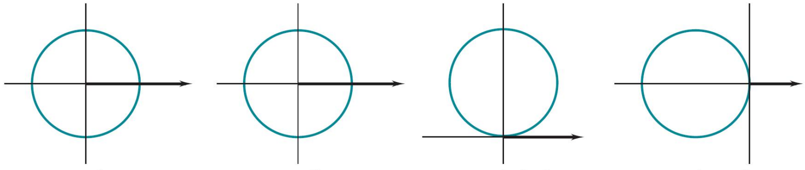

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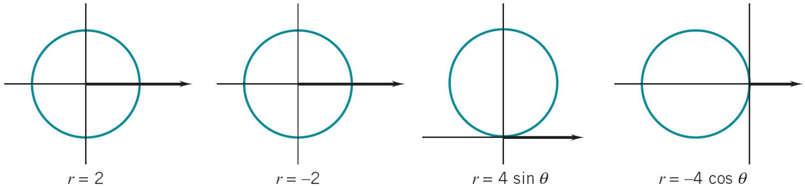

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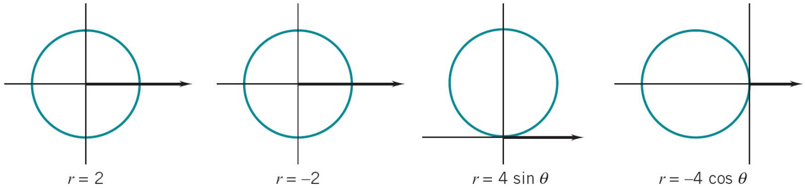

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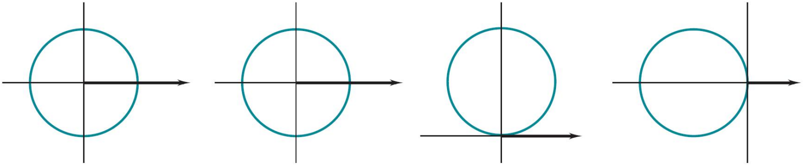


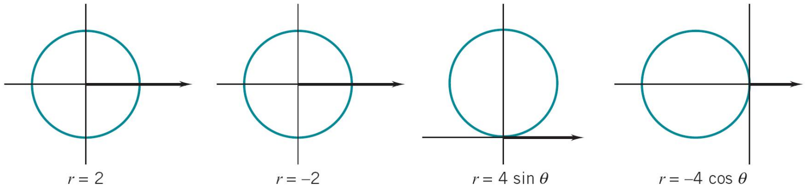
Circle in Polar Coordinates

$$r = 2a \sin \theta, \quad 0 \le \theta \le \pi$$

$$L = \int_0^{\pi} \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{[2a \sin \theta]^2 + [-2a \cos \theta]^2}\, d\theta = 2a \int_0^{\pi} d\theta = 2\pi a$$

Example: Circle of Radius a: $L = 2\pi a$



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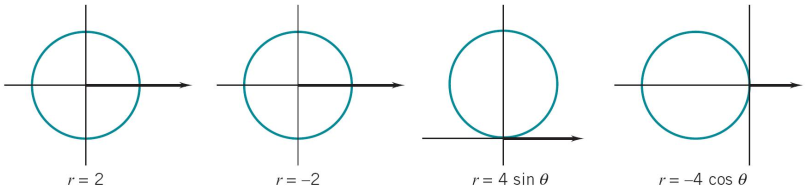

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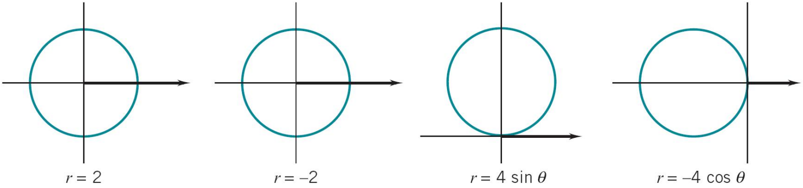

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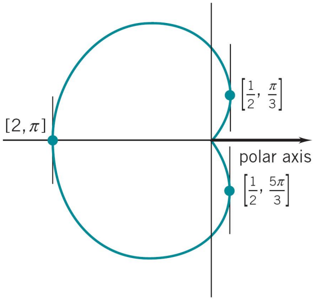

Limaçon: $r = 1 - \cos\theta$

The length of the cardioid: $r = 1 - \cos\theta$, $\theta \in [0, 2\pi]$, is given

$$\int_0^{2\pi} \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2}\, d\theta = 2\int_0^{\pi} \sqrt{[\sin\theta]^2 + [1 - \cos\theta]^2}\, d\theta$$

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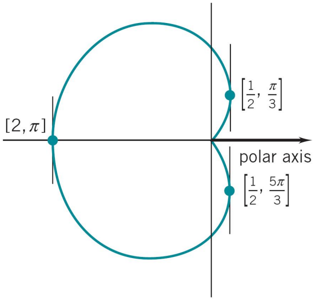

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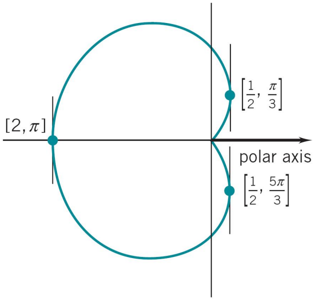

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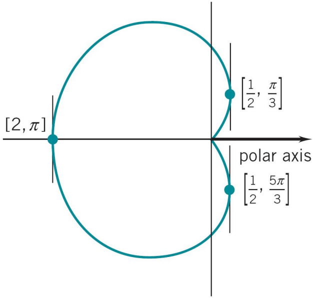

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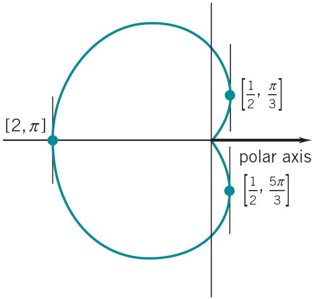

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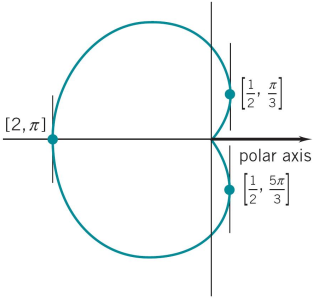

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Example: Logarithmic spiral $r = ae^{b\theta}$

http://scienceblogs.com

Logarithmic Spiral: $r = ae^{b\theta}$

- A logarithmic spiral, equiangular spiral or growth spiral is a special kind of spiral curve which often appears in nature.
- The polar equation of the curve is $r = ae^{b\theta}$ or $\theta = b^{-1}\ln(r/a)$.
- The spiral has the property that the angle ϕ between the tangent and radial line at the point (r, θ) is constant and $\phi = \arctan b^{-1}$.

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Logarithmic Spiral in Motion $r = ae^{-b\theta}$, $\theta \geq 0$

Sprial Motions

- The approach of a hawk to its prey. Their sharpest view is at an angle to their direction of flight.
- The approach of an insect to a light source. They are used to having the light source at a constant angle to their flight path.
- Starting at a point P and moving inward along the spiral with the angle ϕ.

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$$\frac{dr}{d\theta} = -br, \quad r(0) = a, \quad \text{with } b = \cot\alpha.$$

Logarithmic Spiral in Motion $r = ae^{-b\theta}$, $\theta \geq 0$

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Length of the Logarithmic Spiral: $r = ae^{-b\theta}$, $\theta \geq 0$

- The length of the logarithmic spiral: $r = e^{-\theta}$, $\theta \geq 0$, is given

$$L(C) = \int_0^\infty \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2}\, d\theta = \int_0^\infty \sqrt{[e^{-\theta}]^2 + [e^{-\theta}]^2}\, d\theta$$

- The spiral motion $r = ae^{-b\theta}$, $\theta \geq 0$ circles the origin an unbounded number of times without reaching it; yet, the total distance covered on this path is finite:

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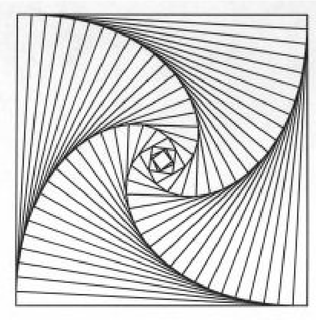
Four Bugs Chasing One Another



Four Bugs Chasing One Another

- Four bugs are at the corners of a square.
- They start to crawl clockwise at a constant rate, each moving toward its neighbor.
- At any instant, they mark the corners of a square. As the bugs get closer to the original square's center, the new square they define rotates and diminishes in size.

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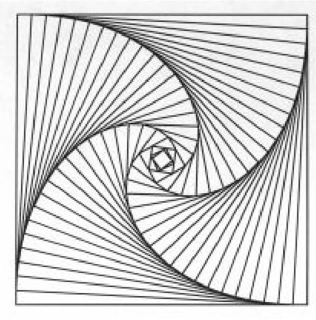

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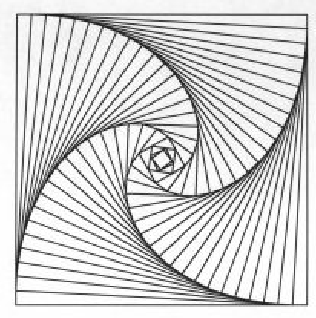

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- They start to crawl clockwise at a constant rate, each moving toward its neighbor.
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- Each bug starts at a corner of the orginal (unit) square that is $1/\sqrt{2}$ away from the origin (i.e., center) and moves inward along the spiral with the angle $\phi = \frac{\pi}{4}$. The spiral motion is described by $\frac{dr}{d\theta} = -r$, $r(0) = 1/\sqrt{2}$.

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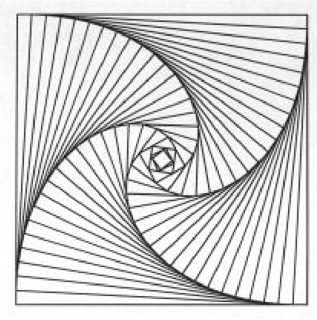

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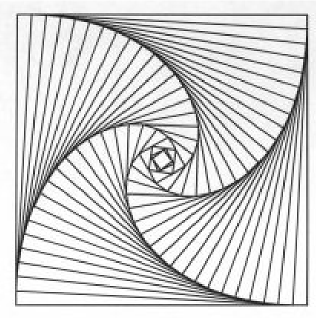

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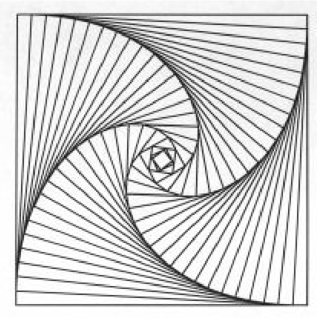

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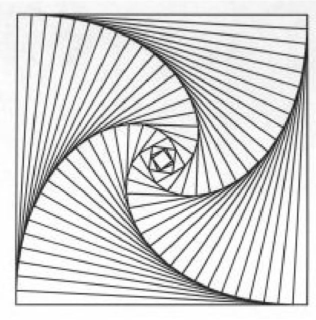

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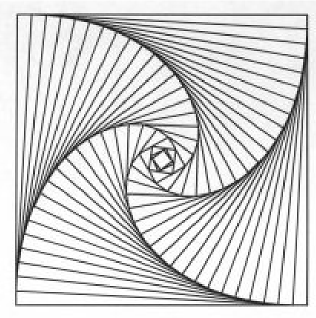

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Outline

- Arc Length
  - Arc Length
  - Examples