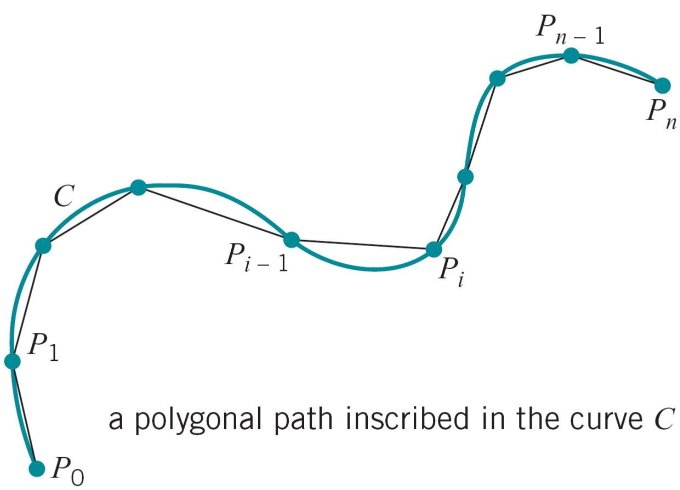
Lecture 16 Section 9.8 Arc Length and Speed

Jiwen He

1 Arc Length

1.1 Arc Length

Arc Length Formulas



Arc Length Formulas

Let $C = \{(x(t), y(t)) : t \in I\}$. [0.5ex] The length of C is

$$L(C) = \int_a^b \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\, dt$$

$$\begin{aligned} d(P_{i-1}, P_i) &= \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2} \\ &= \sqrt{\left[\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}\right]^2 + \left[\frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}}\right]^2}(t_i - t_{i-1}) \\ &= \sqrt{\left[x'(t_i^*)\right]^2 + \left[y'(t_i^*)\right]^2}\Delta t_i \end{aligned}$$

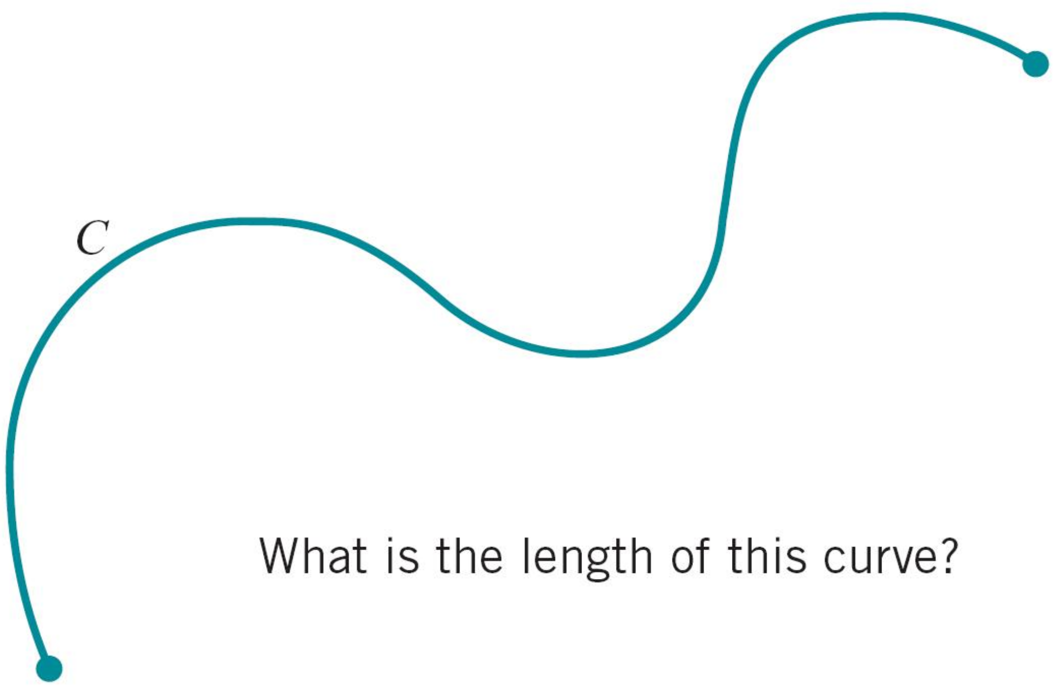
$$\begin{aligned} L(\gamma) &= d(P_0, P_1) + \cdots + d(P_{i-1}, P_i) + \cdots + d(P_{n-1}, P_n) \\ &= \sum_{i=1}^{n} \sqrt{\left[x'(t_i^*)\right]^2 + \left[y'(t_i^*)\right]^2}\Delta t_i \to L(C) \quad \text{as } \Delta t_i \to 0. \end{aligned}$$

Definition 1. • We define the *element of length ds*

$$ds = \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\, dt$$

- The total arc length is
$$L(C) = \int ds = \int_a^b \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2}\, dt$$

Arc Length and Speed Along a Plane Curve



Parametrization by the Motion

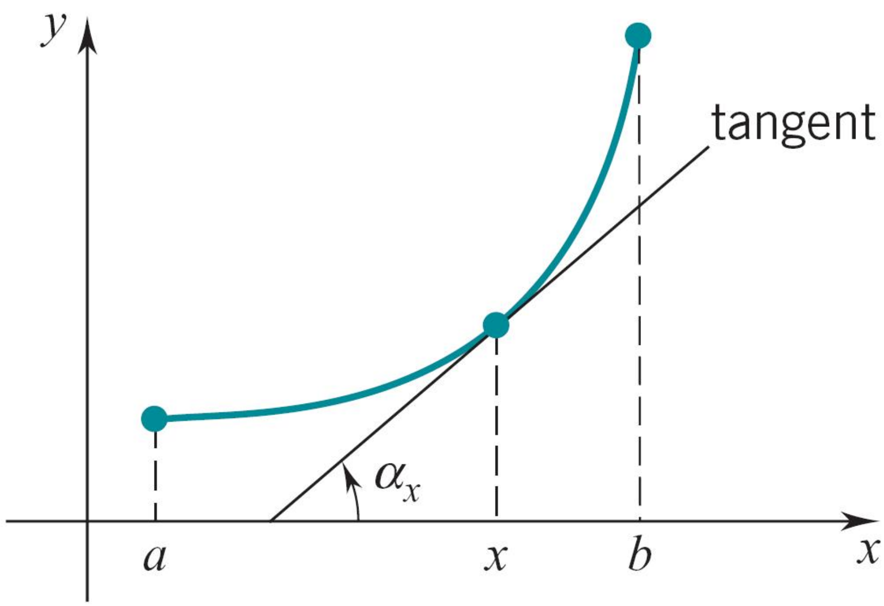
- Imaging an object *moving along the curve* C.
- Let $\mathbf{r}(t) = \big(x(t), y(t)\big)$ the *position* of the object at time t.
- The *velocity* of the object at time t is $\mathbf{v}(t) = \mathbf{r}'(t) = \big(x'(t), y'(t)\big)$.

Arc Length and Speed Along a Plane Curve

- The *speed* of the object at time t is $v(t) = \|\mathbf{v}(t)\| = \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2}$.
- The distance traveled by the object *from time zero to any later time* t is
$$s(t) = \int ds = \int_0^t \sqrt{\big[x'(u)\big]^2 + \big[y'(u)\big]^2}\, du = \int_0^t v(u)\, du.$$
- We have $ds = v(t)\, dt$.

1.2 Examples

Length of the Arc on the Graph of $y = f(x)$



Length of $y = f(x)$, $x \in [a, b]$

The length of the arc on the graph from a to x is

$$s(x) = \int_a^x \sqrt{1 + \left[f'(t)\right]^2}\, dt.$$

$$\Rightarrow \quad ds = \sqrt{1 + \left[f'(x)\right]^2}\, dx.$$

Proof.

Set $x(t) = t$, $y(t) = f(t)$, $t \in [a, b]$. [1ex] Since $x'(t) = 1$, $y'(t) = f'(t)$, then

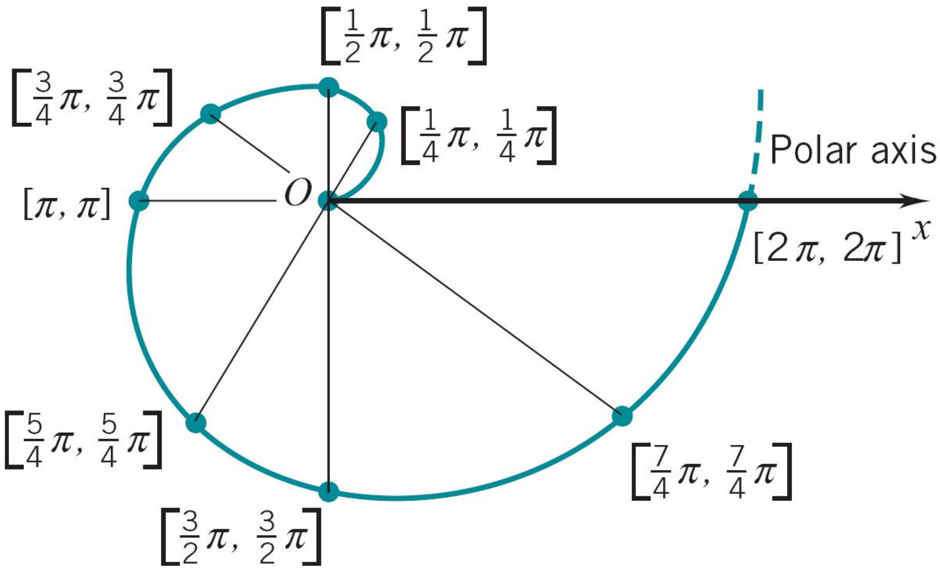
$$s(x) = \int_a^x \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\, dt = \int_a^x \sqrt{1 + \left[f'(t)\right]^2}\, dt.$$

Example 2. • The length of the parabolic arc: $f(x) = x^2$, $x \in [0, 1]$, is given

$$\int_0^1 \sqrt{1 + \left[f'(x)\right]^2}\, dx = \int_0^1 \sqrt{1 + 4x^2}\, dx$$

$$= \left[x\sqrt{\frac{1}{4} + x^2} + \frac{1}{4}\ln(x + \sqrt{\frac{1}{4} + x^2})\right]_0^1 = \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(2 + \sqrt{5}).$$

Length of the Arc on the Graph of $r = \rho(\theta)$



$r = \theta, \quad \theta \geq 0$

spiral of Archimedes

Length of $r = \rho(\theta)$, $\theta \in [\alpha, \beta]$

The length of the arc on the graph from α to θ is

$$s(\theta) = \int_{\alpha}^{\theta} \sqrt{\left[\rho(t)\right]^2 + \left[\rho'(t)\right]^2}\, dt.$$

$$\Rightarrow \quad ds(\theta) = \sqrt{\left[\rho(\theta)\right]^2 + \left[\rho'(\theta)\right]^2}\, d\theta$$

Proof.

Set $x(t) = \rho(t)\cos t$, $y(t) = \rho(t)\sin t$, $t \in [\alpha, \beta]$. [1ex] Since $\left[x'(t)\right]^2 + \left[y'(t)\right]^2 = \left[\rho(t)\right]^2 + \left[\rho'(t)\right]^2$, then

$$s(\theta) = \int_{\alpha}^{\theta} \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\, dt = \int_{\alpha}^{\beta} \sqrt{\left[\rho(t)\right]^2 + \left[\rho'(t)\right]^2}\, dt.$$

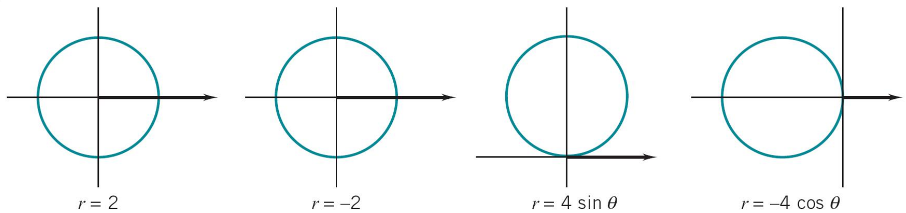
Spiral of Archimedes: $r = \theta$, $\theta \geq 0$

- The length of the arc: $r = \theta$, $\theta \in [0, 2\pi]$, is given

$$\int_0^{2\pi} \sqrt{\left[\rho(\theta)\right]^2 + \left[\rho'(\theta)\right]^2}\, d\theta = \int_0^{2\pi} \sqrt{1 + \theta^2}\, d\theta$$

$$= \left[\frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln(\theta + \sqrt{1+\theta^2})\right]_0^{2\pi} = \cdots$$

Example: Circle of Radius a: $L = 2\pi a$



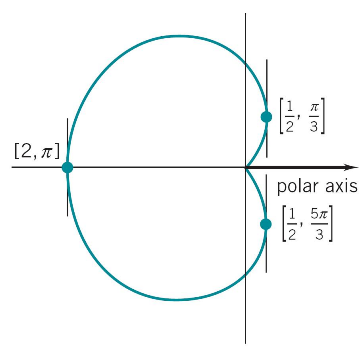
Circle in Polar Coordinates

$$r = a, \quad 0 \le \theta \le 2\pi$$
$$r = 2a\sin\theta, \quad 0 \le \theta \le \pi$$

$$L = \int_0^{2\pi} \sqrt{\big[\rho(\theta)\big]^2 + \big[\rho'(\theta)\big]^2}\, d\theta = \int_0^{2\pi} \sqrt{a^2 + 0}\, d\theta = 2\pi a$$

$$\begin{aligned} L &= \int_0^{\pi} \sqrt{\big[\rho(\theta)\big]^2 + \big[\rho'(\theta)\big]^2}\, d\theta \\ &= \int_0^{\pi} \sqrt{\big[2a\sin\theta\big]^2 + \big[-2a\cos\theta\big]^2} d\theta = 2a \int_0^{\pi} d\theta = 2\pi a \end{aligned}$$

Example: Limaçon



Limaçon: $r = 1 - \cos\theta$

The length of the cardioid: $r = 1 - \cos\theta$, $\theta \in [0, 2\pi]$, is given

$$\begin{aligned} &\int_0^{2\pi} \sqrt{\big[\rho(\theta)\big]^2 + \big[\rho'(\theta)\big]^2}\, d\theta = 2\int_0^{\pi} \sqrt{\big[\sin\theta\big]^2 + \big[1 - \cos\theta\big]^2}\, d\theta \\ &= 2\int_0^{\pi} \sqrt{2(1 - \cos\theta)}\, d\theta = 2\int_0^{\pi} 2\sin\frac{1}{2}\theta\, d\theta = 8\Big[-\cos\frac{1}{2}\theta\Big]_0^{\pi} = 8 \end{aligned}$$

Example: Logarithmic spiral $r = ae^{b\theta}$

`http://scienceblogs.com`

Logarithmic Spiral: $r = ae^{b\theta}$

- A *logarithmic spiral*, *equiangular spiral* or *growth spiral* is a special kind of spiral curve which often appears in nature.
- The polar equation of the curve is $r = ae^{b\theta}$ or $\theta = b^{-1}\ln(r/a)$.
- The spiral has the property that the *angle* ϕ between the tangent and radial line at the point (r, θ) is *constant* and $\phi = \arctan b^{-1}$.

Logarithmic Spiral in Motion $r = ae^{-b\theta}$**,** $\theta \geq 0$

Sprial Motions

- The approach of a hawk to its prey. Their sharpest view is at an angle to their direction of flight.
- The approach of an insect to a light source. They are used to having the light source at a constant angle to their flight path.
- Starting at a point P and moving inward along the spiral with the angle ϕ. Let a be the straight-line distance from P to the origin. The spiral motion is described by

  $$\frac{dr}{d\theta} = -b\,r, \quad r(0) = a, \qquad \text{with } b = \cot\phi.$$

  The polar equation of the path is

  $$r = ae^{-b\theta}, \quad \theta \geq 0$$

Length of the Logarithmic Spiral: $r = ae^{-b\theta}$**,** $\theta \geq 0$

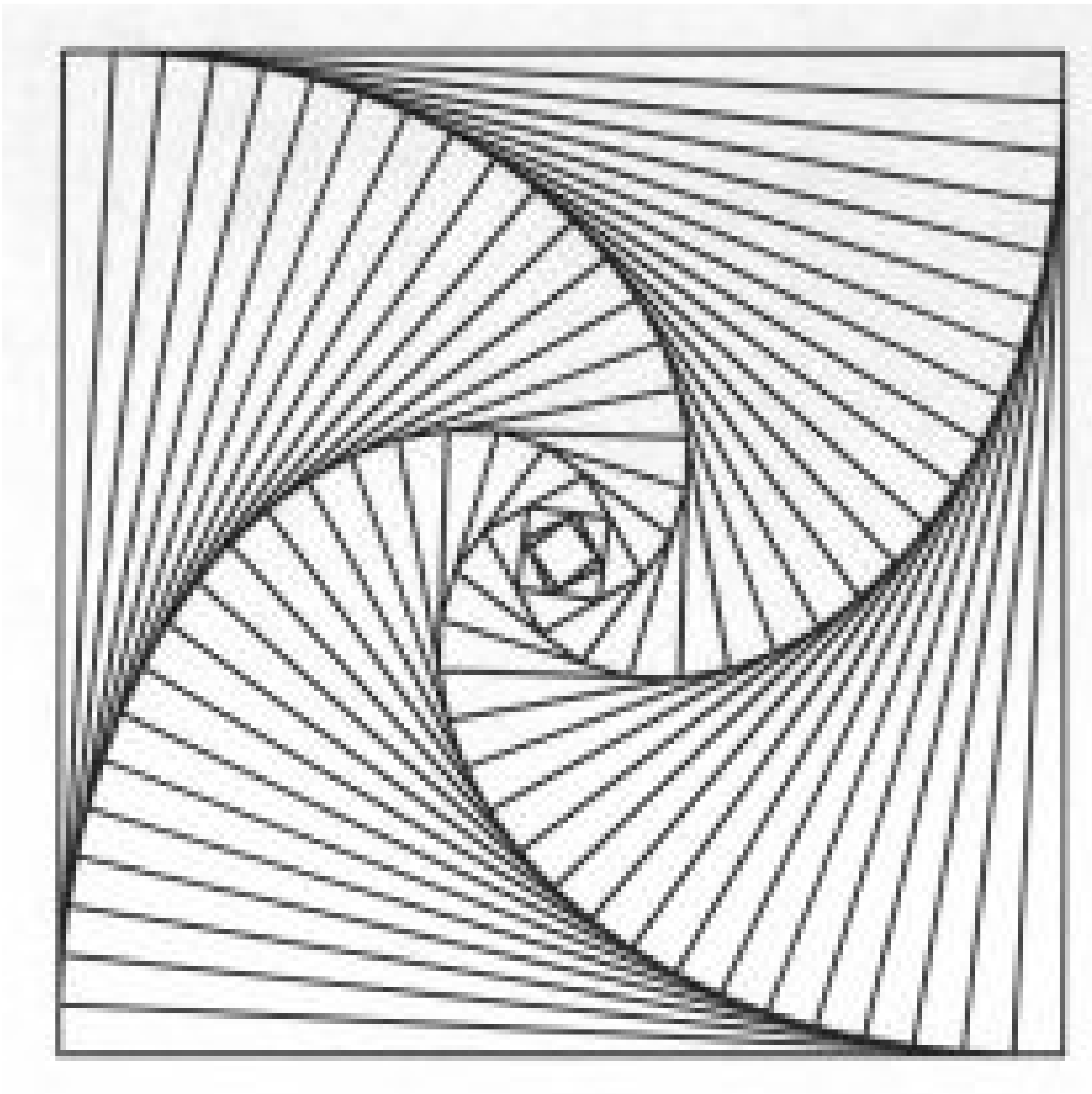
- The length of the logarithmic spiral: $r = e^{-\theta}$, $\theta \geq 0$, is given

  $$L(C) = \int_0^\infty \sqrt{\big[\rho(\theta)\big]^2 + \big[\rho'(\theta)\big]^2}\, d\theta = \int_0^\infty \sqrt{\big[e^{-\theta}\big]^2 + \big[e^{-\theta}\big]^2}\, d\theta$$
  $$= \sqrt{2}\int_0^\infty e^{-\theta}\, d\theta = \sqrt{2}\big[-e^{-\theta}\big]_0^\pi = \sqrt{2}$$

- The spiral motion $r = ae^{-b\theta}$, $\theta \geq 0$ circles the origin an unbounded number of times without reaching it; yet, the total distance covered on this path is finite:

  $$L(C) = \int_0^\infty ds = a/\cos(\phi), \qquad \text{with } \phi = \cot^{-1} b.$$

Four Bugs Chasing One Another



Four Bugs Chasing One Another

- Four bugs are at the corners of a square.
- They start to crawl clockwise at a constant rate, each moving toward its neighbor.
- At any instant, they mark the corners of a square. As the bugs get closer to the original square's center, the new square they define rotates and diminishes in size.

- Each bug starts at a corner of the orginal (unit) square that is $1/\sqrt{2}$ away from the origin (i.e., center) and moves inward along the spiral with the angle $\phi = \frac{\pi}{4}$. The spiral motion is described by $\frac{dr}{d\theta} = -r$, $r(0) = 1/\sqrt{2}$. The polar equation of the path is $r = 1/\sqrt{2}e^{-\theta}$, $\theta \geq 0$. The total distance covered on its path is $L(C) = 1$.

Outline

Contents

1 Arc Length **1**

1.1 Arc Length 1

1.2 Examples 3