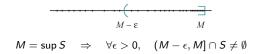
Lecture 17

Section 10.1 Least Upper Bound Axiom Section 10.2 Sequences of Real Numbers

Jiwen He

Department of Mathematics, University of Houston

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Basic Properties of \mathbb{R} : \mathbb{R} being Ordered

Classification

- $\mathbb{N} = \{0, 1, 2, \ldots\} = \{\text{natural numbers}\}$
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots, \} = \{\text{integers}\}$
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• $\mathbb{R} = \{$ real numbers $\} = \mathbb{Q} \cup \{$ irrational numbers $(\pi, \sqrt{2}, \ldots)\}$

$\mathbb R$ is An Ordered Field

- $x \leq y$ and $y \leq z \Rightarrow x \leq z$.
- $x \le y$ and $y \le x$ \Leftrightarrow x = y.
- $\forall x, y \in \mathbb{R} \quad \Rightarrow \quad x \leq y \text{ or } y \leq x.$
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Real Numbers Sequences Review LUB

Archimedean Property and Dedekind Cut Axiom

Archimedean Property

Dedekind Cut Axiom

Least Upper Bound Theorem

Jiwen He, University of Houston

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Real Numbers Sequences Review LUB

Archimedean Property

$$\forall x > 0, \forall y > 0, \exists n \in \mathbb{N} \text{ such that } nx > y.$$

Dedekind Cut Axiom

Let E and F be two nonempty subsets of \mathbb{R} such that

- $E \cup F = \mathbb{R};$
- $E \cap F = \emptyset;$
- $\forall x \in E, \forall y \in F$, we have $x \leq y$.

Then, $\exists z \in \mathbb{R}$ such that

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Every nonempty subset S of $\mathbb R$ with an upper bound has a least upper bound (also called supremum).

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Real NumbersSequencesReviewLUBBasic Properties of \mathbb{R} :Least Upper Bound Property

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Let S be a nonempty subset S of \mathbb{R} .

- S is bounded above if ∃M ∈ ℝ such that x ≤ M for all x ∈ S;
 M is called an upper bound for S.
- S is bounded below if ∃m ∈ ℝ such that x ≥ m for all x ∈ S;
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• S is bounded if it is bounded above and below.

Least Upper Bound Theorem

Every nonempty subset S of \mathbb{R} with an upper bound has a least upper bound (also called supremum)

Proof.

Let $F = \{ upper bounds for S \}$ and $E = \mathbb{R} \setminus E \implies (E, F)$ is a Dedekind cut $\Rightarrow \exists b \in \mathbb{R}$ such that $x \leq b$, $\forall x \in E$ and $b \leq y$. $\forall y \in F$: *b* is also an upper bound of $S \implies b$ is the lub of S.



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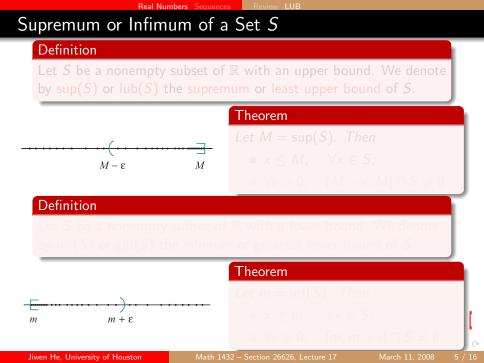
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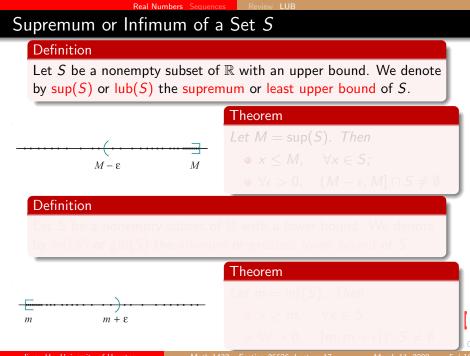
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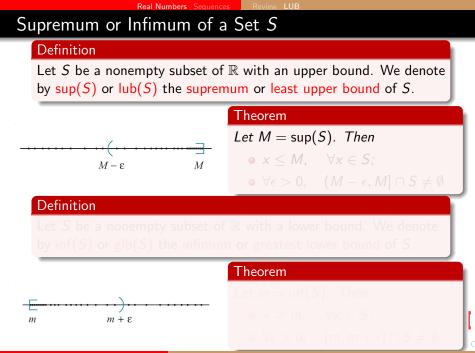




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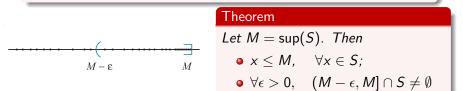
Real Numbers Sequences Supremum or Infimum of a Set S Definition Let S be a nonempty subset of \mathbb{R} with an upper bound. We denote by $\sup(S)$ or $\lim(S)$ the supremum or least upper bound of S. Theorem Let $M = \sup(S)$. Then • x < M. $\forall x \in S$: $M - \epsilon$ M• $\forall \epsilon > 0$, $(M - \epsilon, M] \cap S \neq \emptyset$ Definition Theorem

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Real Numbers Sequences Supremum or Infimum of a Set S

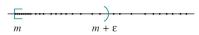
Definition

Let S be a nonempty subset of \mathbb{R} with an upper bound. We denote by $\sup(S)$ or $\lim(S)$ the supremum or least upper bound of S.



Definition





Theorem

Let
$$m = \inf(S)$$
. Then

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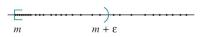
•
$$x \leq M$$
, $\forall x \in S$;

•
$$\forall \epsilon > 0$$
, $(M - \epsilon, M] \cap S \neq \emptyset$

Definition

Let S be a nonempty subset of \mathbb{R} with a lower bound. We denote by inf(S) or glb(S) the infimum or greatest lower bound of S.

Theorem





Real Numbers Sequences Supremum or Infimum of a Set S Definition Let S be a nonempty subset of \mathbb{R} with an upper bound. We denote by $\sup(S)$ or $\lim(S)$ the supremum or least upper bound of S.



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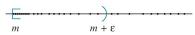
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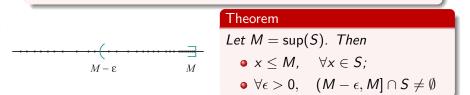
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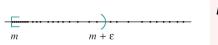
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Real Numbers Sequences Review LUB Supremum or Infimum of a Set S Definition Let S be a nonempty subset of \mathbb{R} with an upper bound. We denote by sup(S) or lub(S) the supremum or least upper bound of S.



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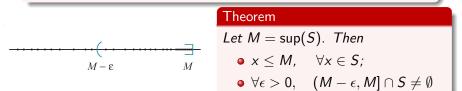
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Examples: Supremum or Infimum of a Set S

Real Numbers Sequences Review LUB

Examples

- Every finite subset of $\mathbb R$ has both upper and lower bounds: $\sup\{1,2,3\}=3, \ \inf\{1,2,3\}=1.$
- If a < b, then
 - $b = \sup[a, b] = \sup[a, b)$ and $a = \inf[a, b] = \inf(a, b]$.
- If $S = \{q \in \mathbb{Q} : e < q < \pi\}$, then inf S = e, sup $S = \pi$.
- If $S=\{x\in\mathbb{R}:x^2<\pi\}$, then inf $S=-\sqrt{3}$, sup $S=\sqrt{3}$

• If $S = \{x \in \mathbb{Q} : x^2 < \pi\}$, then inf $S = -\sqrt{3}$, sup $S = \sqrt{3}$.

Theorem

The notions of infimum and supremum are dual in the sense that $\inf(S) = -\sup(-S)$

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March 11, 2008

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Examples: Supremum or Infimum of a Set S

Real Numbers Sequences Review LUB

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Real Numbers Sequences Review LUB

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March 11, 2008

Definition

A sequence of real numbers is a real-valued function defined on the set of positive integers \mathbb{N}^* :

$$\mathbb{N}^* = \{1, 2, \ldots\}
i n \mapsto a_n = f(n) \in \mathbb{R}.$$

where the n^{th} term f(n) is denoted by a_n .

The sequence a_1, a_2, \ldots , is denoted by $(a_n)_{n=1}^{\infty}$ or (a_n) .

Examples

- $a_n = \frac{1}{n}, n \in \mathbb{N}^*$, is the sequence 1, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$;
- $a_n = \frac{n}{n+1}$, $n \in \mathbb{N}^*$, is the sequence $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, ...
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- $a_n = \cos n\pi = (-1)^n$, $n \in \mathbb{N}^*$, is the sequence -



Real Numbers Sequences

Sequences: Definition

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Real Numbers Sequences

Limit of a Sequence

Definition

Examples



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March 11, 2008

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Real Numbers Sequences

Limit of a Sequence

Definition

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. A real number L is a limit of $(a_n)_{n=1}^{\infty}$, denoted by

$$L=\lim_{n\to\infty}a_n,$$

 $\forall \epsilon > 0, \quad \exists N \in \mathbb{N} \text{ such that } |a_n - L| < \epsilon, \forall n > N.$

Examples

if

• If
$$a_n = \frac{1}{n+1}$$
, $n \in \mathbb{N}^*$, then $\lim_{n \to \infty} a_n = 0$.

 $\frac{1}{N} < \epsilon$. Then, if n > N, we have $0 < \frac{1}{n+1} < \frac{1}{N+1}$

• If $a_n=rac{n}{n+1}$, $n\in\mathbb{N}^*$, then $\lim_{n o\infty}a_n=1$

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For any $\epsilon > 0$ given, choose $N > 0$ such that $\epsilon N > 1$,
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Examples

If a_n = 1/(n+1), n ∈ N*, then lim_{n→∞} a_n = 0. For any ε > 0 given, choose N > 0 such that εN > 1, i.e., 1/N < ε. Then, if n > N, we have 0 < 1/(n+1) < 1/(N+1) < ε.
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Real Numbers Sequences

Convergent Sequence



• A sequence that has a limit is said to be convergent.

• A sequence that has no limit is said to be divergent.

Uniqueness of Limit

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$, then L = M.

Proof.

 $\begin{aligned} \forall \epsilon > 0, \ \exists N > 0 \ \text{such that} \ |a_n - L| < \frac{\epsilon}{2} \ \text{and} \ |a_n - M| < \frac{\epsilon}{2}, \ \forall n > N, \\ \Rightarrow |L - M| \le |a_n - L| + |a_n - M| < \epsilon \quad \Rightarrow \quad L = M. \end{aligned}$

Example

• If $a_n = \cos n\pi = (-1)^n$, $n \in \mathbb{N}^*$, the sequence (a_n) is divergent.

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March 11, 2008

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in the interval $(x - \frac{1}{2}, x + \frac{1}{2})$ has a length $\frac{2}{2}$



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- A sequence that has a limit is said to be convergent.
- A sequence that has no limit is said to be divergent.

Uniqueness of Limit

If
$$\lim_{n o \infty} a_n = L$$
 and $\lim_{n o \infty} a_n = M$, then $L = M$

Proof.

$$\begin{array}{l} \forall \epsilon > 0, \; \exists N > 0 \; \text{such that} \; |a_n - L| < \frac{\epsilon}{2} \; \text{and} \; |a_n - M| < \frac{\epsilon}{2}, \; \forall n > N. \\ \Rightarrow \; |L - M| \leq |a_n - L| + |a_n - M| < \epsilon \; \Rightarrow \; L = M. \end{array}$$

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- If $a_n = \cos n\pi = (-1)^n$, $n \in \mathbb{N}^*$, the sequence (a_n) is divergent.
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Boundedness of a Sequence

Definition

A sequence $(a_n)_{n=1}^{\infty}$ is bounded above or bounded below or bounded if the set $S = \{a_1, a_2, \ldots\}$ is bounded above or bounded below or bounded.

Examples

- If $a_n = \frac{1}{n+1}$, $n \in \mathbb{N}^*$, then the sequence (a_n) is bounded above by $M \ge 1$ and bounded below by $m \le 0$.
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Math 1432 - Section 26626, Lecture 17

0 / 16

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Math 1432 - Section 26626, Lecture 17

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Theorem

Every unbounded sequence is divergent.

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Math 1432 - Section 26626, Lecture 17

March 11, 2008

10 / 16

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 - If $a_n = n^2$, $n \in \mathbb{N}^*$, then (a_n) is increasing, but unbounded above, therefore is divergent.



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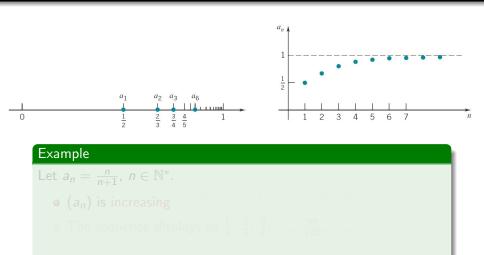
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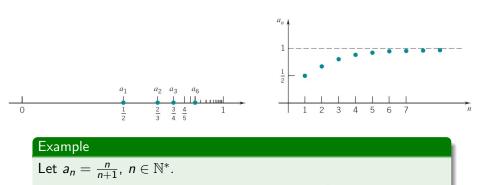
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Math 1432 - Section 26626, Lecture 1

12 / 16

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Example



- (a_n) is increasing $\leftarrow \frac{a_{n+1}}{a_n} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n} > 1$
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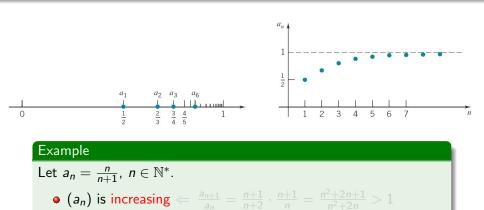
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Math 1432 – Section 26626, Lecture 17

12 / 16

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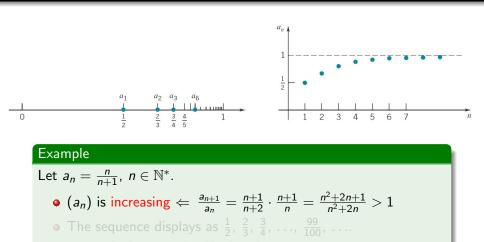
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Math 1432 – Section 26626, Lecture 17

2 / 16

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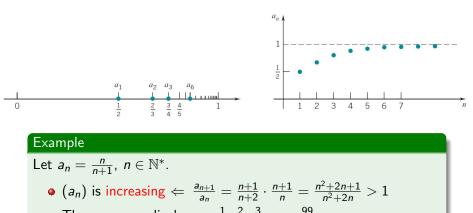


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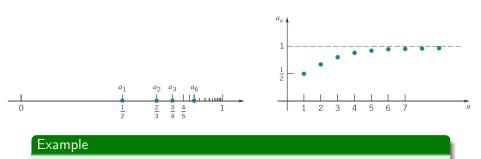


• The sequence displays as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, ..., $\frac{99}{100}$, $\Rightarrow \sup(a_n) = 1$ and $\inf(a_n) = \frac{1}{2}$

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Example



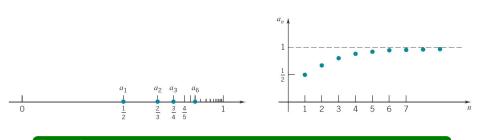
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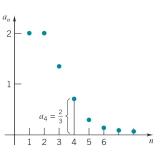
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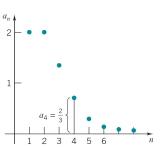
Let $a_n = \frac{2^n}{n}$ with $n = n(n-1)\cdots 1$ • (a_n) is decreasing

• $sup(a_n) = 2$ and $inf(a_n) = 0$

$$\Rightarrow \lim_{n \to \infty} a_n = \inf(a_n) = 0.$$

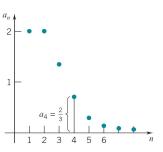


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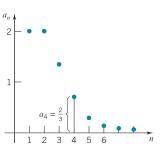
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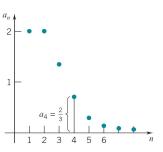
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Example Let $a_n = \frac{2^n}{n}$ with $n = n(n-1)\cdots 1$. • (a_n) is decreasing $\Leftarrow \frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)2^n} = \frac{2}{n+1} < 1$ • $\sup(a_n) = 2$ and $\inf(a_n) = 0$ $\Rightarrow \lim_{n \to \infty} a_n = \inf(a_n) = 0$.





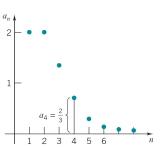
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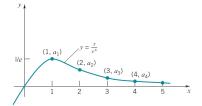


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Example

Let
$$a_n = \frac{n}{e^n}$$
.

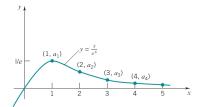
$$\leftarrow \quad \text{Let } f(\mathbf{x}) = \underset{e}{\overset{\times}{\cdot}}.$$

•
$$\sup(a_n) = \frac{1}{e}$$
 and $\inf(a_n) = 0$.

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Example

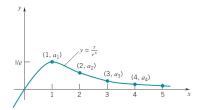
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• (*a_n*) is decreasing

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$$\sup(a_n) = \frac{1}{e}$$
 and $\inf(a_n) = 0$.

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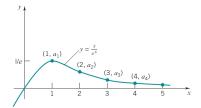
 $\lim_{n\to\infty}a_n=\inf(a_n)=0.$

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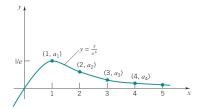
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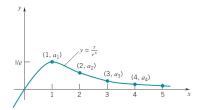
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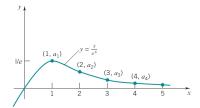
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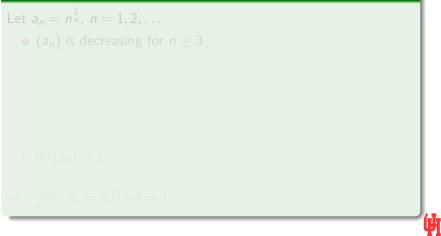
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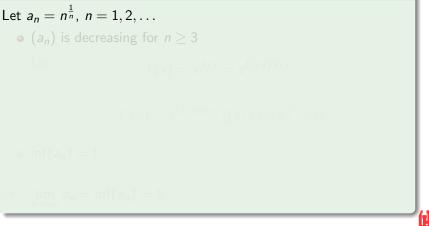
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Outline

- Real Numbers
 - Review
 - Least Upper Bound

• Sequences of Real Numbers



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16 / 16