# Lecture 17 <br> Section 10.1 Least Upper Bound Axiom Section 10.2 Sequences of Real Numbers 

## Jiwen He

Department of Mathematics, University of Houston
jiwenhe@math.uh.edu
http://math.uh.edu/~jiwenhe/Math1432

$M=\sup S \quad \Rightarrow \quad \forall \epsilon>0, \quad(M-\epsilon, M] \cap S \neq \emptyset$

## Basic Properties of $\mathbb{R}: \mathbb{R}$ being Ordered

## Classification

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\text { - } \mathbb{N}=\{0,1,2, \ldots\}=\{\text { natural numbers }\}
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## Archimedean Property and Dedekind Cut Axiom

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## Least Upper Bound Theorem

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## Supremum or Infimum of a Set $S$

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Let $S$ be a nonempty subset of $\mathbb{R}$ with an upper bound. We denote by sup(S) or lub(S) the supremum or least upper bound of $S$

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## Examples: Supremum or Infimum of a Set $S$

## Examples

- Every finite subset of $\mathbb{R}$ has both upper and lower bounds: $\sup \{1,2,3\}=3, \inf \{1,2,3\}=1$
- If $a<b$, then $b=\sup [a, b]=\sup [a, b)$ and $a=\inf [a, b]=\inf (a, b]$


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- If $S=\{q \in \mathbb{Q}: e<q<\pi\}$, then $\inf S=e$, sup $S=\pi$.
- If $S=\left\{x \in \mathbb{R}: x^{2}<\pi\right\}$, then $\inf S=-\sqrt{3}$, sup $S=\sqrt{3}$.


## Theorem

The notions of infimum and supremum are dual in the sense that
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- If $S=\{q \in \mathbb{Q}: e<q<\pi\}$, then inf $S=e$, $\sup S=\pi$.
- If $S=\left\{x \in \mathbb{R}: x^{2}<\pi\right\}$, then $\inf S=-\sqrt{3}$, sup $S=\sqrt{3}$.
- If $S=\left\{x \in \mathbb{Q}: x^{2}<\pi\right\}$, then $\inf S=-\sqrt{3}$, sup $S=\sqrt{3}$.


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The notions of infimum and supremum are dual in the sense that $\inf (S)=-\sup (-S)$
where $-S=\{-s \mid s \in S\}$

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- If $S=\left\{x \in \mathbb{R}: x^{2}<\pi\right\}$, then $\inf S=-\sqrt{3}$, sup $S=\sqrt{3}$.
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## Theorem

The notions of infimum and supremum are dual in the sense that

$$
\inf (S)=-\sup (-S)
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where $-S=\{-s \mid s \in S\}$.

## Sequences: Definition

Definition
A sequence of real numbers is a real-valued function defined on the set of positive integers $\mathbb{N}^{*}$ :

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\mathbb{N}^{*}-\{1,2, \ldots\} \ni n \mapsto a_{n}=f(n) \in \mathbb{R} .
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Uniqueness of Limit
If $\lim _{n \rightarrow \infty} a_{n}=L$ and $\lim _{n \rightarrow \infty} a_{n}=M$, then $L=M$.

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```
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## Boundedness of a Sequence

## Definition

A sequence $\left(a_{n}\right)_{n=1}^{\infty}$ is bounded above or bounded below or bounded if the set $S=\left\{a_{1}, a_{2}, \ldots\right\}$ is bounded above or bounded below or bounded.

Examples

- If $a_{n}=\frac{1}{n+1}, n \in \mathbb{N}^{*}$, then the sequence $\left(a_{n}\right)$ is bounded above by $M \geq 1$ and bounded below by $m \leq 0$.
- If $a_{n}=\cos n \pi=(-1)^{n}, n \in \mathbb{N}^{*}$, then $M \geq 1$ is an upper bound for the sequence $\left(a_{n}\right)$ and $m \leq-1$ is an lower bound for the sequence $\left(a_{n}\right)$.


## Theorem

Every convergent sequence is bounded.

## Theorem

Every unbounded sequence is divergent.

## Monotonic Sequence

## Definition

- A sequence $\left(a_{n}\right)_{n=1}^{\infty}$ is increasing if $a_{n} \leq a_{n+1}$ for all $n \in \mathbb{N}^{*}$


Theorem

Examples


## Monotonic Sequence

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- A sequence $\left(a_{n}\right)_{n=1}^{\infty}$ is decreasing if $a_{n} \geq a_{n+1}$ for all $n \in \mathbb{N}^{*}$


## Theorem

## - A bounded, increasing sequence converges to its lub;

## Examples

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## - a bounded, decreasing sequence converges to its g/b.

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## Examples

- If $a_{n}=\frac{1}{n+1}, n \in \mathbb{N}^{*}$, then $\left(a_{n}\right)$ is decreasing, bounded, and $\lim _{n \rightarrow \infty} a_{n}=\inf \left(a_{n}\right)=0$.


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\lim _{n \rightarrow \infty} a_{n}=\inf \left(a_{n}\right)=0 . \Leftarrow \frac{a_{n+1}}{a_{n}}=\frac{1}{n+2} \frac{n+1}{1}=\frac{n+1}{n+2}<1
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above, therefore is divergent.

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- A bounded, increasing sequence converges to its lub;
- a bounded, decreasing sequence converges to its glb.


## Examples

- If $a_{n}=\frac{1}{n+1}, n \in \mathbb{N}^{*}$, then $\left(a_{n}\right)$ is decreasing, bounded, and $\lim _{n \rightarrow \infty} a_{n}=\inf \left(a_{n}\right)=0 . \Leftarrow \frac{a_{n+1}}{a_{n}}=\frac{1}{n+2} \frac{n+1}{1}=\frac{n+1}{n+2}<1$
- If $a_{n}=n^{2}, n \in \mathbb{N}^{*}$, then $\left(a_{n}\right)$ is increasing, but unbounded above, therefore is divergent.


## Example




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Let $a_{n}=\frac{n}{n+1}, n \in \mathbb{N}^{*}$.

- $\left(a_{n}\right)$ is increasing


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- $\left(a_{n}\right)$ is increasing $\Leftarrow \frac{a_{n+1}}{a_{n}}=\frac{n+1}{n+2} \cdot \frac{n+1}{n}=\frac{n^{2}+2 n+1}{n^{2}+2 n}>1$
- The sequence displays as $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{99}{100}$


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\Rightarrow \sup \left(a_{n}\right)=1 \text { and } \inf \left(a_{n}\right)=\frac{1}{2}
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$\Rightarrow \quad \lim _{n \rightarrow \infty} a_{n}=\sup \left(a_{n}\right)=1$.


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Let $a_{n}=\frac{2^{n}}{n}$ with $n=n(n-1) \cdots 1$. - $\left(a_{n}\right)$ is decreasing

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\Leftarrow \frac{a_{n+1}}{a_{n}}=\frac{2^{n+1}}{(n+1)} \frac{n}{2^{n}}=\frac{2}{n+1}<1
$$

$$
\text { - } \sup \left(a_{n}\right)=2 \text { and } \inf \left(a_{n}\right)=0
$$

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\begin{aligned}
& \text { Let } a_{n}=\frac{n}{e^{n}} . \\
& \quad\left(a_{n}\right) \text { is decreasing }
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$\Leftarrow$ Let $f(x)=\frac{x}{e^{x}}$.



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Let $a_{n}=\frac{n}{e^{n}}$.

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$\Leftarrow$ Let $f(x)=\frac{x}{e^{x}}$.

- $\sup \left(a_{n}\right)=\frac{1}{e}$ and $\inf \left(a_{n}\right)=0$


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Let $a_{n}=\frac{n}{e^{n}}$.

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$$
\Leftarrow \quad \text { Let } f(x)=\frac{x}{e^{x}} \text {. }
$$

$$
f^{\prime}(x)=\frac{e^{x}-x e^{x}}{e^{2 x}}=\frac{1-x}{e^{x}}<0
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\begin{aligned}
& \text { Let } a_{n}=n^{\frac{1}{n}}, n=1,2, \ldots \\
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$$

- $\inf \left(a_{n}\right)=1$
$\square$


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## Outline

- Real Numbers
- Review
- Least Upper Bound
- Sequences of Real Numbers

