Lecture 17

Section 10.1 Least Upper Bound Axiom
Section 10.2 Sequences of Real Numbers

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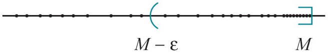

$$M = \sup S \quad \Rightarrow \quad \forall \epsilon > 0, \quad (M - \epsilon, M] \cap S \neq \emptyset$$

Basic Properties of $\mathbb{R}$: $\mathbb{R}$ being Ordered

Classification

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- $x \leq y$ and $y \leq z \quad \Rightarrow \quad x \leq z.$
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- $\forall x, y \in \mathbb{R} \quad \Rightarrow \quad x \leq y$ or $y \leq x.$
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Archimedean Property and Dedekind Cut Axiom

Archimedean Property

$\forall x > 0, \forall y > 0, \exists n \in \mathbb{N}$ such that $nx > y$.

Dedekind Cut Axiom

Let E and F be two nonempty subsets of $\mathbb{R}$ such that

- $E \cup F = \mathbb{R}$;
- $E \cap F = \emptyset$;
- $\forall x \in E, \forall y \in F$, we have $x \leq y$.

Then, $\exists z \in \mathbb{R}$ such that

$$x \leq z, \quad \forall x \in E \qquad \text{and} \qquad z \leq y, \quad \forall y \in F.$$

Least Upper Bound Theorem

Every nonempty subset S of $\mathbb{R}$ with an upper bound has a least upper bound (also called supremum).

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- S is bounded if it is bounded above and below.

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Proof.

Let $F = \{\text{upper bounds for } S\}$ and $E = \mathbb{R} \setminus F$ $\Rightarrow$ (E, F) is a Dedekind cut $\Rightarrow$ $\exists b \in \mathbb{R}$ such that $x \leq b$, $\forall x \in E$ and $b \leq y$, $\forall y \in F$; b is also an upper bound of S $\Rightarrow$ b is the lub of S.

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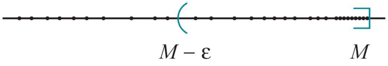
Supremum or Infimum of a Set S

Definition

Let S be a nonempty subset of $\mathbb{R}$ with an upper bound. We denote by $\sup(S)$ or $\text{lub}(S)$ the supremum or least upper bound of S.



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Let $M = \sup(S)$. Then

- $x \leq M, \quad \forall x \in S;$
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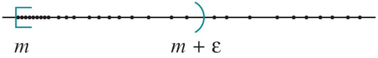

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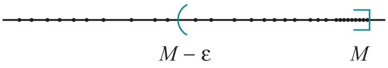

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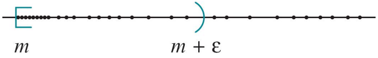

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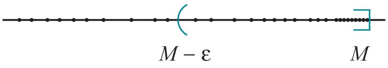

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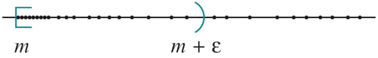

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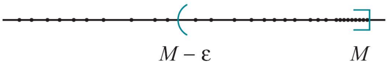

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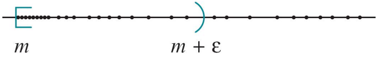

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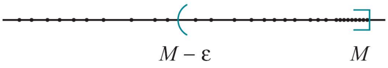

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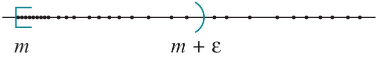

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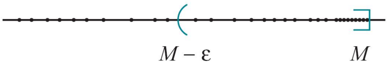

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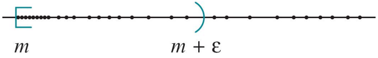

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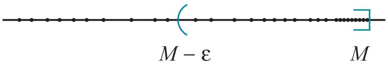

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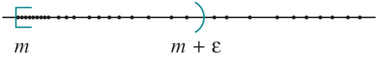

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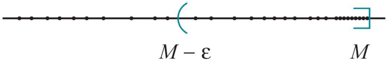

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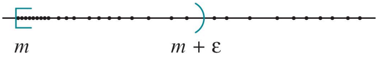

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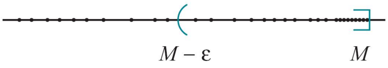

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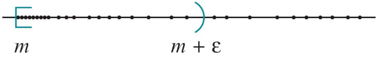

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Examples: Supremum or Infimum of a Set S

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$$\mathbb{N}^* = \{1, 2, \dots\} \ni n \mapsto a_n = f(n) \in \mathbb{R}.$$

where the n^{th} term $f(n)$ is denoted by a_n.

The sequence a_1, a_2, $\dots$, is denoted by $(a_n)_{n=1}^{\infty}$ or (a_n).

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- $a_n = \cos n\pi = (-1)^n$, $n \in \mathbb{N}^*$, is the sequence -1, 1, -1, ...

Limit of a Sequence

Definition

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. A real number L is a limit of $(a_n)_{n=1}^{\infty}$, denoted by

$$L = \lim_{n\to\infty} a_n,$$

if

$$\forall \epsilon > 0, \quad \exists N \in \mathbb{N} \text{ such that } |a_n - L| < \epsilon, \forall n > N.$$

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- If $a_n = \frac{1}{n+1}$, $n \in \mathbb{N}^*$, then $\lim_{n\to\infty} a_n = 0$.

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- A sequence that has a limit is said to be convergent.
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Uniqueness of Limit

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$, then $L = M$.

Proof.

$\forall \epsilon > 0$, $\exists N > 0$ such that $|a_n - L| < \frac{\epsilon}{2}$ and $|a_n - M| < \frac{\epsilon}{2}$, $\forall n > N$.

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A sequence $(a_n)_{n=1}^{\infty}$ is bounded above or bounded below or bounded if the set $S = \{a_1, a_2, \ldots\}$ is bounded above or bounded below or bounded.

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- *A bounded, increasing sequence converges to its lub;*
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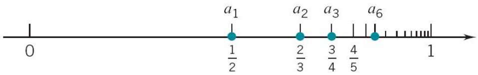

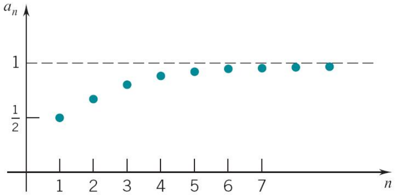

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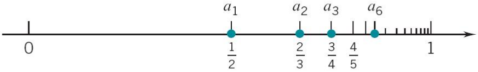

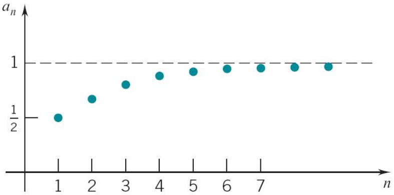

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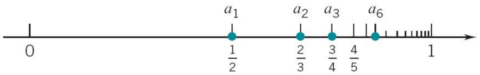

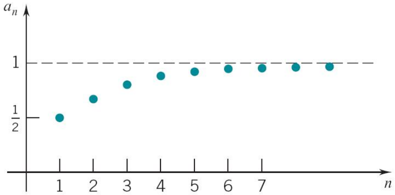

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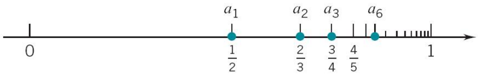

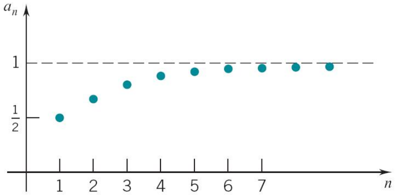

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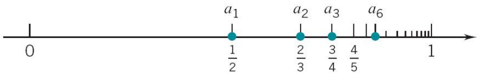

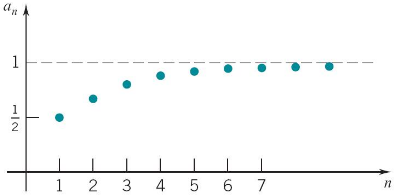

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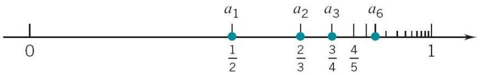

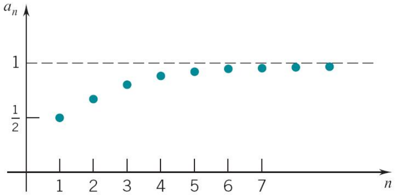

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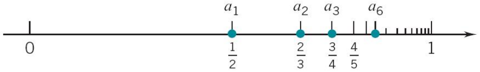

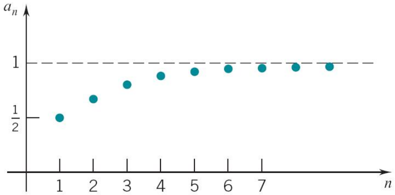

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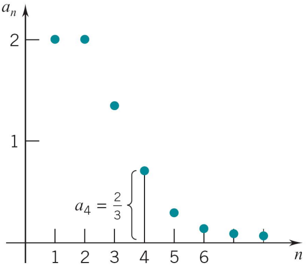

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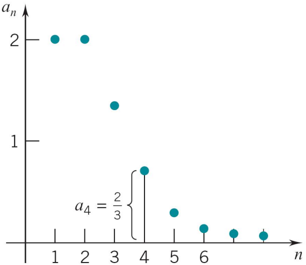

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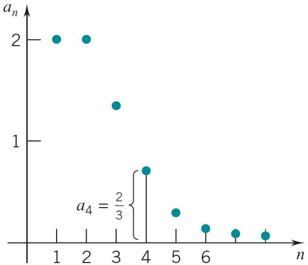

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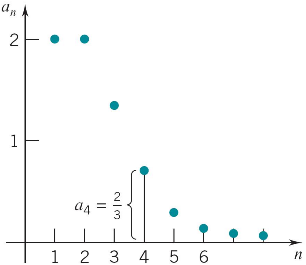

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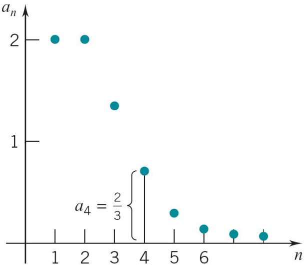

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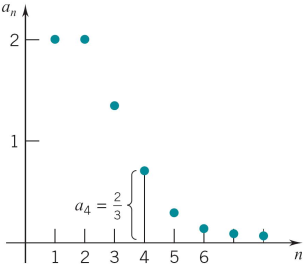

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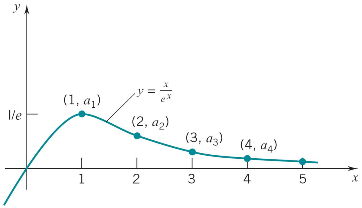

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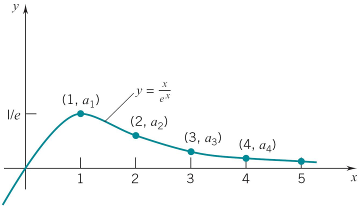

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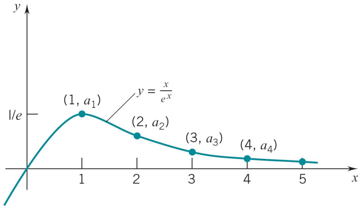

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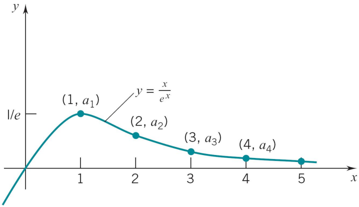

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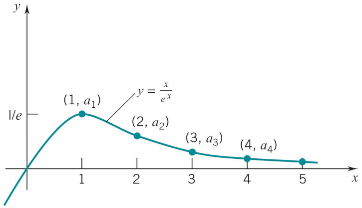

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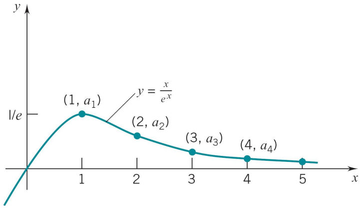

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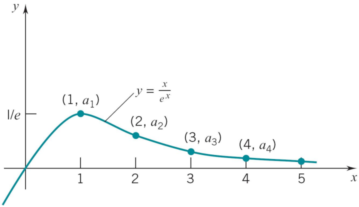

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$\Rightarrow \quad \lim_{n\to\infty} a_n = \inf(a_n) = 1.$

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Outline

- Real Numbers
  - Review
  - Least Upper Bound
- Sequences of Real Numbers