Lecture 17 Section 10.1 Least Upper Bound Axiom Section 10.2 Sequences of Real Numbers

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1 Real Numbers

1.1 Review

Basic Properties of $\mathbb{R}$: $\mathbb{R}$ being Ordered

Classification

- $\mathbb{N} = \{0, 1, 2, \ldots\} = \{\text{natural numbers}\}$
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots, \} = \{\text{integers}\}$
- $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0\} = \{\text{rational numbers}\}$
- $\mathbb{R} = \{\text{real numbers}\} = \mathbb{Q} \cup \{\text{irrational numbers } (\pi, \sqrt{2}, \ldots)\}$

$\mathbb{R}$ is An Ordered Field

- $x \leq y$ and $y \leq z \quad \Rightarrow \quad x \leq z.$
- $x \leq y$ and $y \leq x \quad \Leftrightarrow \quad x = y.$
- $\forall x, y \in \mathbb{R} \quad \Rightarrow \quad x \leq y$ or $y \leq x.$
- $x \leq y$ and $z \in \mathbb{R} \quad \Rightarrow \quad x + z \leq y + z.$
- $x \geq 0$ and $y \geq 0 \quad \Rightarrow \quad xy \geq 0.$

Archimedean Property and Dedekind Cut Axiom

Archimedean Property

$$\forall x > 0,\ \forall y > 0,\ \exists n \in \mathbb{N} \text{ such that } nx > y.$$

Dedekind Cut Axiom

Let E and F be two nonempty subsets of $\mathbb{R}$ such that

- $E \cup F = \mathbb{R}$;

- $E \cap F = \emptyset$;
- $\forall x \in E$, $\forall y \in F$, we have $x \leq y$.

Then, $\exists z \in \mathbb{R}$ such that

$$x \leq z, \quad \forall x \in E \qquad \text{and} \qquad z \leq y, \quad \forall y \in F.$$

Least Upper Bound Theorem
Every nonempty subset S of $\mathbb{R}$ with an upper bound has a *least upper bound* (also called *supremum*).

1.2 Least Upper Bound

Basic Properties of $\mathbb{R}$: Least Upper Bound Property

Definition 1. Let S be a nonempty subset S of $\mathbb{R}$.

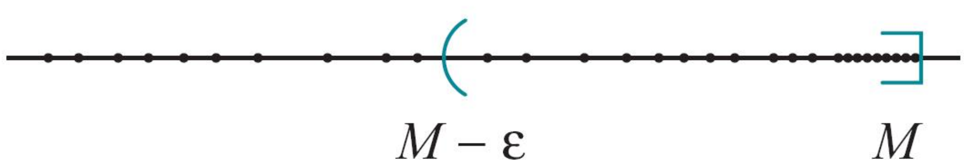
- S is *bounded above* if $\exists M \in \mathbb{R}$ such that $x \leq M$ for all $x \in S$; M is called an *upper bound* for S.
- S is *bounded below* if $\exists m \in \mathbb{R}$ such that $x \geq m$ for all $x \in S$; m is called an *lower bound* for S.
- S is *bounded* if it is bounded above and below.

Least Upper Bound Theorem
Every nonempty subset S of $\mathbb{R}$ with an upper bound has a *least upper bound* (also called *supremum*).

Proof.
Let $F = \{\text{upper bounds for } S\}$ and $E = \mathbb{R} \setminus E \quad \Rightarrow \quad (E, F)$ is a Dedekind cut $\Rightarrow \quad \exists b \in \mathbb{R}$ such that $x \leq b$, $\forall x \in E$ and $b \leq y$, $\forall y \in F$; b is also an upper bound of $S \quad \Rightarrow \quad b$ is the lub of S.

Supremum or Infimum of a Set S

Definition 2. Let S be a nonempty subset of $\mathbb{R}$ with an upper bound. We denote by $\sup(S)$ or *lub*(S) the *supremum* or *least upper bound* of S.

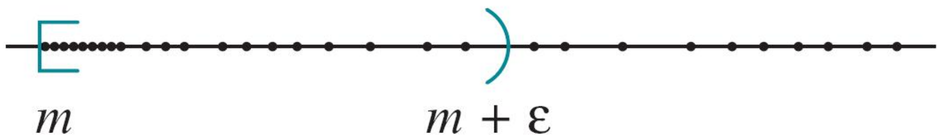


Theorem 3. *Let $M = \sup(S)$. Then*

- $x \leq M, \quad \forall x \in S;$

- $\forall \epsilon > 0, \quad (M-\epsilon, M] \cap S \neq \emptyset$

Definition 4. Let S be a nonempty subset of $\mathbb{R}$ with a lower bound. We denote by $\inf(S)$ or $glb(S)$ the *infimum* or *greatest lower bound* of S.



Theorem 5. *Let $m = \inf(S)$. Then*

- $x \geq m, \quad \forall x \in S;$
- $\forall \epsilon > 0, \quad [m, m+\epsilon] \cap S \neq \emptyset$

Examples: Supremum or Infimum of a Set S

Examples 6.
- Every finite subset of $\mathbb{R}$ has both upper and lower bounds: $\sup\{1,2,3\} = 3$, $\inf\{1,2,3\} = 1$.
- If $a < b$, then $b = \sup[a,b] = \sup[a,b)$ and $a = \inf[a,b] = \inf(a,b]$.
- If $S = \{q \in \mathbb{Q} : e < q < \pi\}$, then inf $S = e$, sup $S = \pi$.
- If $S = \{x \in \mathbb{R} : x^2 < \pi\}$, then inf $S = -\sqrt{3}$, sup $S = \sqrt{3}$.
- If $S = \{x \in \mathbb{Q} : x^2 < \pi\}$, then inf $S = -\sqrt{3}$, sup $S = \sqrt{3}$.

Theorem 7. *The notions of infimum and supremum are dual in the sense that*

$$\inf(S) = -\sup(-S)$$

where $-S = \{-s | s \in S\}$.

2 Sequences of Real Numbers

Sequences: Definition

Definition 8. A *sequence of real numbers* is a real-valued function defined on the set of positive integers $\mathbb{N}^*$:

$$\mathbb{N}^* = \{1, 2, \ldots\} \ni n \mapsto a_n = f(n) \in \mathbb{R}.$$

where the n^{th} term $f(n)$ is denoted by a_n. [2ex] The sequence a_1, a_2, …, is denoted by $(a_n)_{n=1}^{\infty}$ or (a_n).

Examples 9.
- $a_n = \frac{1}{n}$, $n \in \mathbb{N}^*$, is the sequence 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, …;
- $a_n = \frac{n}{n+1}$, $n \in \mathbb{N}^*$, is the sequence $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, …;
- $a_n = n^2$, $n \in \mathbb{N}^*$, is the sequence 1, 4, 9, 16, …;
- $a_n = \cos n\pi = (-1)^n$, $n \in \mathbb{N}^*$, is the sequence -1, 1, -1, ….

Limit of a Sequence

Definition 10. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. A real number L is a *limit of* $(a_n)_{n=1}^{\infty}$, denoted by

$$L = \lim_{n \to \infty} a_n,$$

if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|a_n - L| < \epsilon \; \forall n > N$.

Examples 11.

- If $a_n = \frac{1}{n+1}$, $n \in \mathbb{N}^*$, then $\lim_{n\to\infty} a_n = 0$. For any $\epsilon > 0$ given, choose $N > 0$ such that $\epsilon N > 1$, i.e., $\frac{1}{N} < \epsilon$. Then, if $n > N$, we have $0 < \frac{1}{n+1} < \frac{1}{N+1} < \epsilon$.
- If $a_n = \frac{n}{n+1}$, $n \in \mathbb{N}^*$, then $\lim_{n\to\infty} a_n = 1$. For any $\epsilon > 0$ given, choose $N > 0$ such that $\epsilon N > 1$, i.e., $\frac{1}{N} < \epsilon$. Then, if $n > N$, we have $0 < \left|\frac{n}{n+1} - 1\right| = \frac{1}{n} < \frac{1}{N} < \epsilon$.

Convergent Sequence

Definition 12.

- A sequence that has a limit is said to be *convergent*.
- A sequence that has no limit is said to be *divergent*.

Uniqueness of Limit

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$, then $L = M$.

Proof.

$\forall \epsilon > 0$, $\exists N > 0$ such that $|a_n - L| < \frac{\epsilon}{2}$ and $|a_n - M| < \frac{\epsilon}{2}$, $\forall n > N$. $\Rightarrow$ $|L - M| \le |a_n - L| + |a_n - M| < \epsilon \quad \Rightarrow \quad L = M$.

Example 13.

- If $a_n = \cos n\pi = (-1)^n$, $n \in \mathbb{N}^*$, the sequence (a_n) is *divergent*. If $\epsilon = \frac{1}{3}$, then the interval $(x - \frac{1}{3}, x + \frac{1}{3})$ has a length $\frac{2}{3}$ that is < 1; $\forall x \in \mathbb{R}$, it can not contain 1 and -1 at the same time. Therefore it is not possible to find N such that $|a_n - x| < \frac{1}{3}$ if $n > N$.

Boundedness of a Sequence

Definition 14. A sequence $(a_n)_{n=1}^{\infty}$ is *bounded above* or *bounded below* or *bounded* if the set $S = \{a_1, a_2, \ldots\}$ is *bounded above* or *bounded below* or *bounded*.

Examples 15.

- If $a_n = \frac{1}{n+1}$, $n \in \mathbb{N}^*$, then the sequence (a_n) is *bounded above* by $M \ge 1$ and *bounded below* by $m \le 0$.
- If $a_n = \cos n\pi = (-1)^n$, $n \in \mathbb{N}^*$, then $M \ge 1$ is an *upper bound* for the sequence (a_n) and $m \le -1$ is an *lower bound* for the sequence (a_n).

Theorem 16. *Every convergent sequence is bounded.*

$\lim_{n\to\infty} a_n = L \Rightarrow \forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_n - L| < \epsilon$, $\forall n > N \Rightarrow |a_n| \le |a_n - L| + |L| < \epsilon + |L|$, $\forall n > N \Rightarrow$ done!

Theorem 17. *Every unbounded sequence is divergent.*

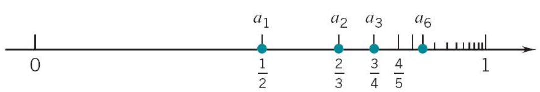
Monotonic Sequence

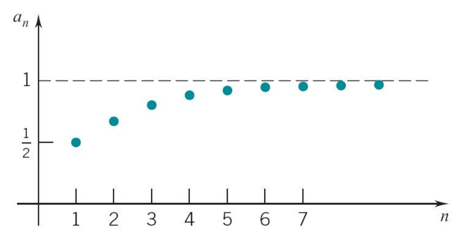
Definition 18.
- A sequence $(a_n)_{n=1}^{\infty}$ is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}^*$.
- A sequence $(a_n)_{n=1}^{\infty}$ is *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}^*$.

Theorem 19.
- *A bounded, increasing sequence converges to its lub;*
- *a bounded, decreasing sequence converges to its glb.*

Examples 20.
- If $a_n = \frac{1}{n+1}$, $n \in \mathbb{N}^*$, then (a_n) is *decreasing*, *bounded*, and $\lim_{n\to\infty} a_n = \inf(a_n) = 0$. $\Leftarrow$ $\frac{a_{n+1}}{a_n} = \frac{1}{n+2}\frac{n+1}{1} = \frac{n+1}{n+2} < 1$
- If $a_n = n^2$, $n \in \mathbb{N}^*$, then (a_n) is *increasing*, but *unbounded above*, therefore is *divergent*.

Example



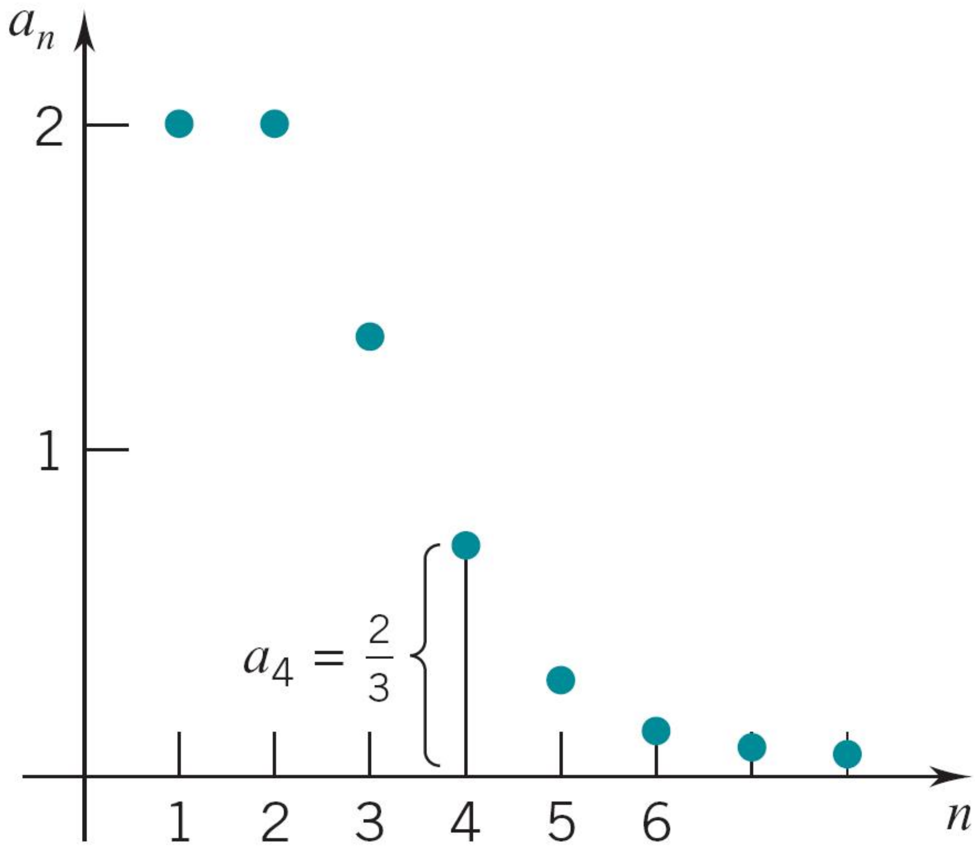


Example 21. Let $a_n = \frac{n}{n+1}$, $n \in \mathbb{N}^*$.

- (a_n) is *increasing* $\Leftarrow$ $\frac{a_{n+1}}{a_n} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n} > 1$
- The sequence displays as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\ldots$, $\frac{99}{100}$, $\ldots$. $\Rightarrow$ $\sup(a_n) = 1$ and $\inf(a_n) = \frac{1}{2}$

$\Rightarrow$ $\lim\limits_{n\to\infty} a_n = \sup(a_n) = 1.$

Example

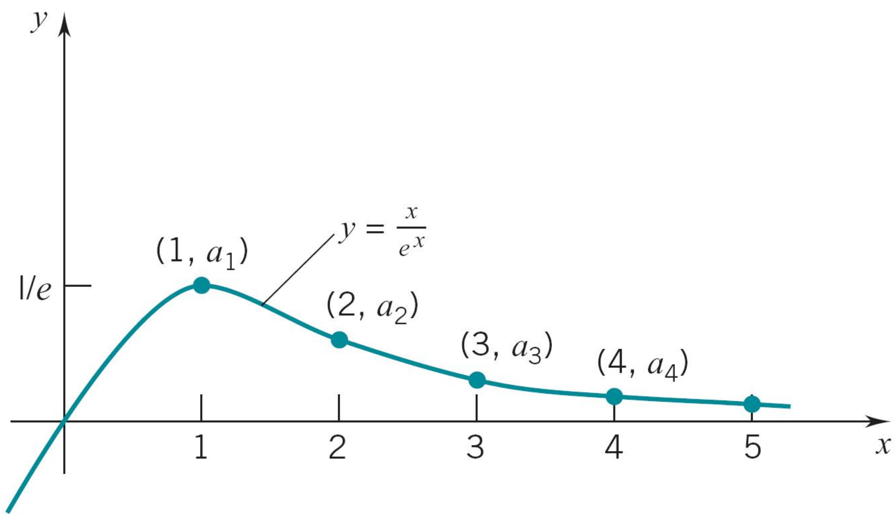


Example 22. Let $a_n = \frac{2^n}{n}$ with $n\!=n(n-1)\cdots 1$.

- (a_n) is decreasing $\;\Leftarrow\;$ $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}}{(n+1)}\dfrac{n}{2^n} = \dfrac{2}{n+1} < 1$
- $\sup(a_n) = 2$ and $\inf(a_n) = 0$

$\Rightarrow \quad \lim_{n\to\infty} a_n = \inf(a_n) = 0.$

Example



Example 23. Let $a_n = \frac{n}{e^n}$.

- (a_n) is decreasing[4ex] $\Leftarrow$ Let $f(x) = \frac{x}{e^x}$. [1ex]
$$f'(x) = \frac{e^x - xe^x}{e^{2x}} = \frac{1-x}{e^x} < 0$$
- $\sup(a_n) = \frac{1}{e}$ and $\inf(a_n) = 0$.

$\Rightarrow \lim_{n\to\infty} a_n = \inf(a_n) = 0.$

Example

Example 24. Let $a_n = n^{\frac{1}{n}}$, $n = 1, 2, \ldots$

- (a_n) is decreasing for $n \geq 3$ [1ex] Let
$$f(x) = x^{1/x} = e^{(1/x)\ln x}.$$
[1ex]
$$f'(x) = e^{(1/x)\ln x}\big((1/x)\ln x\big)' < 0$$
- $\inf(a_n) = 1$

$\Rightarrow \lim_{n\to\infty} a_n = \inf(a_n) = 1.$

Outline

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