Lecture 18

Section 10.3 Limit of Sequence
Section 10.4 Some Important Limits

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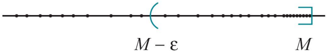

$$M = \sup S \quad \Rightarrow \quad \forall \epsilon > 0, \quad (M - \epsilon, M] \cap S \neq \emptyset$$

Properties of Limits

Properties of Limits: 1

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then

- $\lim_{n\to\infty} c a_n = cL$
- $\lim_{n\to\infty} (a_n + b_n) = L + M$
- $\lim_{n\to\infty} a_n b_n = LM$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ for $M \neq 0$.
- $\lim_{n\to\infty} f(a_n) = f(L)$ for f being continuous at L.

Example

If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} b_n = e^L$ where $b_n = e^{a_n}$.

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Limit of Sequence Defined by Rational Function

Properties of Limits

Let $a_n = \alpha_p n^p + \alpha_{p-1} n^{p-1} + \cdots + \alpha_0$ with $\alpha_p \neq 0$, and $b_n = \beta_q n^q + \beta_{q-1} n^{q-1} + \cdots + \beta_0$ with $\beta_q \neq 0$. Then

- If $p < q$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$.
- If $p = q$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\alpha_p}{\beta_q}$.
- If $p > q$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ does not exist, and the sequence $\dfrac{a_n}{b_n}$ diverges.

Example

$$\lim_{n\to\infty} \frac{3n^4 - 2n^2 + 1}{n^5 - 3n^3} = 0. \qquad \lim_{n\to\infty} \frac{1 - 4n^7}{n^7 + 12n} = -4.$$

$$\lim_{n\to\infty} \frac{n^4 - 3n^2 + n + 2}{n^3 + 7n} \text{ does not exist.}$$

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Pinching Theorem

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Suppose that for all n greater than some integer N,

$$a_n \le b_n \le c_n.$$

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Suppose that $|b_n| \le a_n$, $\forall n > N$ for some N.
If $a_n \to 0$, then $b_n \to 0$.

Example

$$\frac{\cos n}{n} \to 0, \qquad \text{since } \left|\frac{\cos n}{n}\right| \le \frac{1}{n} \text{ and } \frac{1}{n} \to 0.$$

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Some Important Limits: 1

$$\lim_{n\to\infty} \frac{1}{n^\alpha} = 0, \quad \alpha > 0.$$

Proof.

$\forall \alpha > 0$, $\exists p \in \mathbb{N}$ s.t. $1/p < \alpha$. Then

$$0 < \frac{1}{n^\alpha} = \left(\frac{1}{n}\right)^\alpha < \left(\frac{1}{n}\right)^{1/p}$$

Since $\frac{1}{n} \to 0$ and $f(u) = u^{1/p}$ is continuous at 0, we have

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/p} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/p} = \lim_{u\to 0} u^{1/p} = 0^{1/p} = 0.$$

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Some Important Limits: 2

$$\lim_{n\to\infty} x^{\frac{1}{n}} = 1, \quad x > 0.$$

Proof.

Note that $\forall x$,

$$\ln\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\ln x \to 0, \quad \text{as } n \to \infty.$$

Since $f(u) = e^u$ is continuous at 0, we have

$$\lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln x} = \lim_{u\to 0} e^u = e^0 = 1.$$

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$$\lim_{n\to\infty} x^n = 0 \quad \text{if } |x| < 1.$$

(The limit does not exist if $|x| > 1$ or $x = -1$.)

Proof.

Note that for any x s.t. $0 < |x| < 1$, we have $\ln|x| < 0$ and

$$\ln(|x|^n) = n \ln|x| \to -\infty, \qquad \text{as } n \to \infty.$$

Since $\lim_{u\to-\infty} e^u = 0$, we have

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Therefore,

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Let $(a_n)_{n=1}^{\infty}$, $a_n \neq 0$ $\forall n$ such that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$. If $\rho < 1$, then the sequence $(a_n)_{n=1}^{\infty}$ converges to 0, and, if $\rho > 1$, it diverges.

Proof. ($\rho < 1$)

Since $0 < \frac{1+\rho}{2} < 1$, $\lim_{n\to\infty} \left(\frac{1+\rho}{2}\right)^n = 0$. Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$, $\exists N_1$ s.t. $\forall n > N_1$, $\left| \frac{a_{n+1}}{a_n} \right| \leq \rho + \frac{1-\rho}{2}$, thus $|a_{n+1}| \leq \frac{1+\rho}{2}|a_n|$. Then, for $n > N_1$,

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$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Proof.

Note that

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)!}\frac{n!}{|x|^n} = \frac{|x|}{n+1} \to 0 \quad \text{as } n \to \infty$$

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$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^p}{|x|^{n+1}} \frac{|x|^n}{n^p} = \frac{1}{|x|}\left(1+\frac{1}{n}\right) \to \frac{1}{|x|} \quad \text{as } n \to \infty$$

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$$\lim_{n\to\infty} \frac{n^p}{x^n} = 0, \quad |x| > 1.$$

Some Important Limits: 5

$$\lim_{n\to\infty} \frac{n^p}{x^n} = 0, \quad |x| > 1.$$

Proof.

Note that

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^p}{|x|^{n+1}} \frac{|x|^n}{n^p} = \frac{1}{|x|}\left(1+\frac{1}{n}\right) \to \frac{1}{|x|} \quad \text{as } n \to \infty$$

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Some Important Limits: 6

$$\lim_{n \to \infty} \frac{\ln n}{n} = 0.$$

Proof.

$$0 \le \frac{\ln n}{n} = \frac{1}{n}\int_1^n \frac{dt}{t} \le \frac{1}{n}\int_1^n \frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{n}}(\sqrt{n} - 1) \to 0 \text{ as } n \to 0.$$

By the pinching theorem, $\lim_{n \to \infty} \frac{\ln n}{n} = 0.$

$$\lim_{n \to \infty} n^{\frac{1}{n}} = 1.$$

Proof.

$$\lim_{n \to \infty} n^{\frac{1}{n}} = \lim_{n \to \infty} e^{\frac{\ln n}{n}} = \lim_{u \to 0} e^u = e^0 = 1.$$

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Alternative proof is done by L'Hôpital's Rule (10.5),

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{u\to\infty} \frac{\ln u}{u} = \lim_{u\to\infty} \frac{\frac{1}{u}}{1} = 0.$$

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Some Important Limits: 7

$$\lim_{n\to\infty} \frac{\ln n}{n^\alpha} = 0, \quad \alpha > 0.$$

Proof.

By L'Hôpital's Rule (10.5),

$$\lim_{n\to\infty} \frac{\ln n}{n^\alpha} = \lim_{u\to\infty} \frac{\ln u}{u^\alpha} = \lim_{u\to\infty} \frac{u^{-1}}{\alpha u^{\alpha-1}} = \lim_{u\to\infty} \frac{1}{\alpha u^\alpha} = 0.$$

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Some Important Limits: 8

$$\lim_{n\to\infty} n \ln\left(1+\frac{x}{n}\right) = x.$$

Proof.

$$\lim_{n\to\infty} n \ln\left(1+\frac{x}{n}\right) = x \lim_{n\to\infty}\left[\frac{\ln\left(1+\frac{x}{n}\right) - \ln 1}{\frac{x}{n}}\right]$$

$$= x \lim_{h\to 0}\left[\frac{\ln(1+h) - \ln 1}{h}\right]$$

$$= x \lim_{h\to 0}\left[\frac{\ln(u+h) - \ln u}{h}\right]_{u=1}$$

$$= x(\ln u)'_{u=1} = x\frac{1}{u}\Big|_{u=1} = x.$$

Some Important Limits: 8

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Proof.

$$\begin{aligned}
\lim_{n\to\infty} n \ln\left(1+\frac{x}{n}\right) &= x \lim_{n\to\infty}\left[\frac{\ln\left(1+\frac{x}{n}\right)-\ln 1}{\frac{x}{n}}\right] \\
&= x \lim_{h\to 0}\left[\frac{\ln(1+h)-\ln 1}{h}\right] \\
&= x \lim_{h\to 0}\left[\frac{\ln(u+h)-\ln u}{h}\right]_{u=1} \\
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Some Important Limits: 8

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Some Important Limits: 8

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Proof.

Alternative proof is done by L'Hôpital's Rule (10.5),

$$\lim_{n\to\infty} n \ln\left(1+\frac{x}{n}\right) = \lim_{u\to\infty} \frac{\ln(1+x/u)}{1/u} = \lim_{u\to\infty} \frac{1/(1+x/u)(-x/u^2)}{-1/u^2} = x.$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

Proof.

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} e^{n \ln\left(1+\frac{x}{n}\right)} = \lim_{u\to x} e^u = e^x.$$

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Outline

- Limit of Sequence
  - Properties of Limits
- Some Important Limits
  - Some Limits