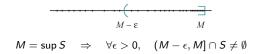
### Lecture 18 Section 10.3 Limit of Sequence Section 10.4 Some Important Limits

#### Jiwen He

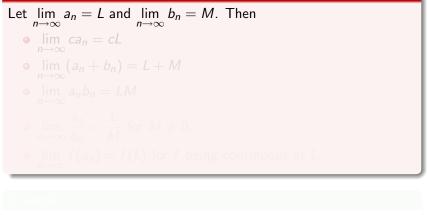
Department of Mathematics, University of Houston

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### Properties of Limits: 1



If  $\lim_{n\to\infty} a_n = L$ , then  $\lim_{n\to\infty} b_n = e^L$  where  $b_n = e^{a_n}$ 



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Let \lim_{n \to \infty} a_n = L and \lim_{n \to \infty} b_n = M. Then

• \lim_{n \to \infty} ca_n = cL

• \lim_{n \to \infty} (a_n + b_n) = L + M

• \lim_{n \to \infty} a_n b_n = LM

• \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{L}{M} for M \neq 0.

• \lim_{n \to \infty} f(a_n) = f(L) for f being continuous A
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Limit of Sequence Some Limits

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• If  $p < q$ , then  $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ .  
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• If  $p > q$ , then  $\lim_{n \to \infty} \frac{a_n}{b_n}$  does not exist, and the sequence  $\frac{a_n}{b_n}$   
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$$\lim_{n \to \infty} \frac{3n^4 - 2n^2 + 1}{n^5 - 3n^3} = 0. \qquad \lim_{n \to \infty} \frac{1 - 4n^7}{n^7 + 12n} = -4.$$
$$\lim_{n \to \infty} \frac{n^4 - 3n^2 + n + 2}{n^3 + 7n} \text{ does not exist.}$$

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### Pinching Theorem

Suppose that for all n greater than some integer N,

 $a_n \leq b_n \leq c_n$ .

If  $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ , then  $\lim_{n\to\infty} b_n = L$ .

Suppose that  $|b_n| \le a_n$ ,  $\forall n > N$  for some N. If  $a_n \to 0$ , then  $b_n \to 0$ .



 $\frac{\cos n}{n} \to 0$ , since  $\left| \frac{\cos n}{n} \right| \le \frac{1}{n}$  and  $\frac{1}{n} \to 0$ .



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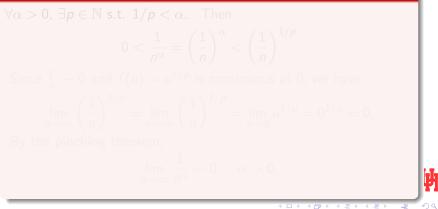
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$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}=0, \quad \alpha>0.$$

### Proof.



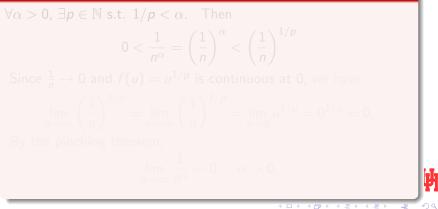
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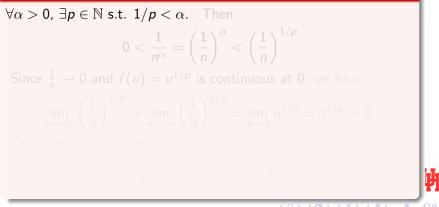
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### Proof.

$$\forall \alpha > 0, \ \exists p \in \mathbb{N} \text{ s.t. } 1/p < \alpha. \text{ Then} \\ 0 < \frac{1}{n^{\alpha}} = \left(\frac{1}{n}\right)^{\alpha} < \left(\frac{1}{n}\right)^{1/p} \\ \text{Since } \frac{1}{n} \to 0 \text{ and } f(u) = u^{1/p} \text{ is continuous at } 0, \text{ we } 1 \\ \lim_{n \to \infty} \left(\frac{1}{n}\right)^{1/p} = \lim_{n \to \infty} \left(\frac{1}{n}\right)^{1/p} = \lim_{u \to 0} u^{1/p} = 0^{1/p} \\ \text{By the pinching theorem,} \\ \lim_{n \to \infty} \frac{1}{n^{\alpha}} = 0, \quad \alpha > 0. \end{cases}$$

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$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}=0, \quad \alpha>0.$$

### Proof.

 $\forall \alpha > 0, \exists p \in \mathbb{N} \text{ s.t. } 1/p < \alpha.$  Then  $0 < \frac{1}{n^{\alpha}} = \left(\frac{1}{n}\right)^{\alpha} < \left(\frac{1}{n}\right)^{1/p}$ Since  $\frac{1}{n} \to 0$  and  $f(u) = u^{1/p}$  is continuous at 0, we have

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$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}=0, \quad \alpha>0.$$

#### Proof.

 $\begin{aligned} \forall \alpha > 0, \ \exists p \in \mathbb{N} \text{ s.t. } 1/p < \alpha. \quad \text{Then} \\ 0 < \frac{1}{n^{\alpha}} = \left(\frac{1}{n}\right)^{\alpha} < \left(\frac{1}{n}\right)^{1/p} \\ \text{Since } \frac{1}{n} \to 0 \text{ and } f(u) = u^{1/p} \text{ is continuous at } 0, \text{ we have} \\ \lim_{n \to \infty} \left(\frac{1}{n}\right)^{1/p} = \lim_{n \to \infty} \left(\frac{1}{n}\right)^{1/p} = \lim_{u \to 0} u^{1/p} = 0^{1/p} = 0. \end{aligned}$ By the pinching theorem,

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}=0, \quad \alpha>0.$$

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$$\lim_{n\to\infty}x^{\frac{1}{n}}=1, \quad x>0.$$

#### Proof.

Note that  $\forall x$ ,  $\ln\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\ln x \to 0$ , as  $n \to \infty$ . Since  $f(u) = e^{u}$  is continuous at 0, we have  $\lim_{n \to \infty} x^{\frac{1}{n}} = \lim_{n \to \infty} e^{\frac{1}{n}\ln x} = \lim_{u \to 0} e^{u} = e^{0} = 1$ .



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#### Proof.

Note that  $\forall x$ ,  $\ln\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\ln x \to 0, \quad \text{as } n \to \infty.$ Since  $f(u) = e^{u}$  is continuous at 0, we have  $\lim_{n \to \infty} x^{\frac{1}{n}} = \lim_{n \to \infty} e^{\frac{1}{n}\ln x} = \lim_{u \to 0} e^{u} = e^{0} = 1.$ 



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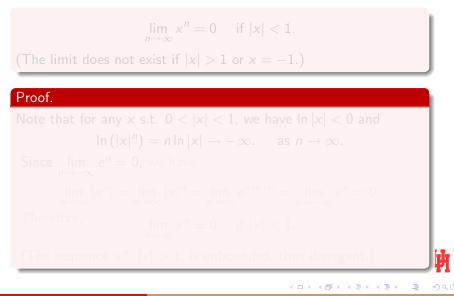
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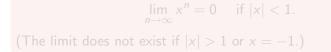
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# Alembert's Rule

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Let 
$$(a_n)_{n=1}^{\infty}$$
,  $a_n \neq 0 \ \forall n$  such that  $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$ . If  $\rho < 1$ , then the sequence  $(a_n)_{n=1}^{\infty}$  converges to 0, and, if  $\rho > 1$ , it diverges.

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Since 
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## Alembert's Rule

### Alembert's Rule

Let 
$$(a_n)_{n=1}^{\infty}$$
,  $a_n \neq 0 \ \forall n$  such that  $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$ . If  $\rho < 1$ , then the sequence  $(a_n)_{n=1}^{\infty}$  converges to 0, and, if  $\rho > 1$ , it diverges.

### Proof. ( $\rho > 1$ )

$$\exists \delta > 0 \text{ s.t. } \rho > 1 + \delta. \text{ Since } \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho, \ \exists N_2 \text{ s.t. } \forall n > N_2, \\ \left| \frac{a_{n+1}}{a_n} \right| \ge \rho - \frac{\delta}{2}. \text{ As } \rho - \frac{\delta}{2} > 1 + \frac{\delta}{2}, \ |a_{n+1}| \ge \left( 1 + \frac{\delta}{2} \right) |a_n|. \text{ Then,} \\ \text{for } n > N_1, \\ |a_n| \ge \left( 1 + \frac{\delta}{2} \right) |a_{n-1}| \ge \left( 1 + \frac{\delta}{2} \right)^2 |a_{n-2}| \ge \cdots \ge \left( 1 + \frac{\delta}{2} \right)^{n-N_2} |a_{N_2}|$$

and therefore  $(a_n)$  is unbounded, thus divergent.

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### Proof.



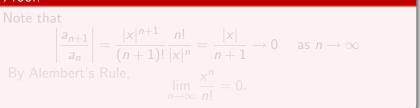


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$$\lim_{n\to\infty}\frac{x^n}{n!}=0.$$

### Proof.





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# Note that $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} = \frac{|x|}{n+1} \to 0 \quad \text{as } n \to \infty$ By Alembert's Rule, $\lim_{n \to \infty} \frac{x^n}{n!} = 0.$



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$$\lim_{n\to\infty}\frac{n^p}{x^n}=0, \quad |x|>1.$$

### Proof.

Note that  

$$\begin{vmatrix} a_{n+1} \\ a_n \end{vmatrix} = \frac{(n+1)^p}{|x|^{n+1}} \frac{|x|^n}{n^p} = \frac{1}{|x|} \left(1 + \frac{1}{n}\right) \to \frac{1}{|x|} \quad \text{as } n \to \infty$$
Since  $\frac{1}{|x|} < 1$ , by Alembert's Rule,  

$$\lim_{n \to \infty} \frac{n^p}{x^n} = 0, \quad |x| > 1.$$



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# Some Important Limits: 6



### Proof.

$$0 \leq \frac{\ln n}{n} = \frac{1}{n} \int_{1}^{n} \frac{dt}{t} \leq \frac{1}{n} \int_{1}^{n} \frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{n}} (\sqrt{n} - 1) \to 0 \text{ as } n \to 0.$$
  
By the pinching theorem, 
$$\lim_{n \to \infty} \frac{\ln n}{n} = 0.$$

$$\lim_{n\to\infty}n^{\frac{1}{n}}=1.$$

### Proof.

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Alternative proof is done by L'Hôpital's Rule (10.5),

$$\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{u\to\infty}\frac{\ln u}{u} = \lim_{u\to\infty}\frac{\frac{1}{u}}{1} = 0.$$

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$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = \lim_{u\to0} e^u = e^0 = 1.$$

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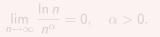
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### Proof.

By L'Hôpital's Rule (10.5),  
$$\lim_{n \to \infty} \frac{\ln n}{n^{\alpha}} = \lim_{u \to \infty} \frac{\ln u}{u^{\alpha}} = \lim_{u \to \infty} \frac{u^{-1}}{\alpha u^{\alpha-1}} = \lim_{u \to \infty} \frac{1}{\alpha u^{\alpha}} = 0.$$



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# Some Important Limits: 7

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$$\lim_{n\to\infty}n\ln\left(1+\frac{x}{n}\right)=x.$$

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$$\lim_{n \to \infty} n \ln\left(1 + \frac{x}{n}\right) = x \lim_{n \to \infty} \left[\frac{\ln\left(1 + \frac{x}{n}\right) - \ln 1}{\frac{x}{n}}\right]$$
$$= x \lim_{h \to 0} \left[\frac{\ln(1+h) - \ln 1}{h}\right]$$
$$= x \lim_{h \to 0} \left[\frac{\ln(u+h) - \ln u}{h}\right]_{u=1}$$
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$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x.$$

Proof.

$$\lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = \lim_{n \to \infty} e^{n \ln \left( 1 + \frac{x}{n} \right)} = \lim_{u \to \infty} e^u = e^x.$$

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# Some Important Limits: 8

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## Outline

Limit of SequenceProperties of Limits

Some Important Limits
 Some Limits

