Lecture 18 Section 10.3 Limit of Sequence Section 10.4 Some Important Limits

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1 Limit of Sequence

1.1 Properties of Limits

Properties of Limits

Properties of Limits: 1
Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then

- $\lim_{n\to\infty} ca_n = cL$
- $\lim_{n\to\infty} (a_n + b_n) = L + M$
- $\lim_{n\to\infty} a_n b_n = LM$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ for $M \neq 0$.
- $\lim_{n\to\infty} f(a_n) = f(L)$ for f being continuous at L.

Example 1. If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} b_n = e^L$ where $b_n = e^{a_n}$.

Limit of Sequence Defined by Rational Function

Properties of Limits
Let $a_n = \alpha_p n^p + \alpha_{p-1} n^{p-1} + \cdots + \alpha_0$ with $\alpha_p \neq 0$, and $b_n = \beta_q n^q + \beta_{q-1} n^{q-1} + \cdots + \beta_0$ with $\beta_q \neq 0$. Then

- If $p < q$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.
- If $p = q$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\alpha_p}{\beta_q}$.
- If $p > q$, then $\lim_{n\to\infty} \frac{a_n}{b_n}$ does not exist, and the sequence $\frac{a_n}{b_n}$ diverges.

Example 2.

$$\lim_{n\to\infty} \frac{3n^4 - 2n^2 + 1}{n^5 - 3n^3} = 0. \qquad \lim_{n\to\infty} \frac{1 - 4n^7}{n^7 + 12n} = -4.$$
$$\lim_{n\to\infty} \frac{n^4 - 3n^2 + n + 2}{n^3 + 7n} \text{ does not exist.}$$

Pinching Theorem
Pinching Theorem
Suppose that for all n greater than some integer N,
$$a_n \leq b_n \leq c_n.$$

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Suppose that $|b_n| \leq a_n$, $\forall n > N$ for some N. If $a_n \to 0$, then $b_n \to 0$.

Example 3.
$$\frac{cosn}{n} \to 0, \quad \text{since } \left|\frac{cosn}{n}\right| \leq \frac{1}{n} \text{ and } \frac{1}{n} \to 0.$$

2 Some Important Limits

2.1 Some Limits

Some Important Limits: 1

$$\lim_{n\to\infty} \frac{1}{n^\alpha} = 0, \quad \alpha > 0.$$

Proof.
$\forall \alpha > 0$, $\exists p \in \mathbb{N}$ s.t. $1/p < \alpha$. Then
$$0 < \frac{1}{n^\alpha} = \left(\frac{1}{n}\right)^\alpha < \left(\frac{1}{n}\right)^{1/p}$$
Since $\frac{1}{n} \to 0$ and $f(u) = u^{1/p}$ is continuous at 0, we have
$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/p} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/p} = \lim_{u\to 0} u^{1/p} = 0^{1/p} = 0.$$
By the pinching theorem,
$$\lim_{n\to\infty} \frac{1}{n^\alpha} = 0, \quad \alpha > 0.$$

Some Important Limits: 2

$$\lim_{n\to\infty} x^{\frac{1}{n}} = 1, \quad x > 0.$$

Proof.
Note that $\forall x$,
$$\ln\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\ln x \to 0, \quad \text{as } n \to \infty.$$
Since $f(u) = e^u$ is continuous at 0, we have
$$\lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln x} = \lim_{u\to 0} e^u = e^0 = 1.$$

Some Important Limits: 3

$$\lim_{n\to\infty} x^n = 0 \quad \text{if } |x| < 1.$$

(The limit does not exist if $|x| > 1$ or $x = -1$.)

Proof.
Note that for any x s.t. $0 < |x| < 1$, we have $\ln |x| < 0$ and

$$\ln (|x|^n) = n \ln |x| \to -\infty, \qquad \text{as } n \to \infty.$$

Since $\lim\limits_{u\to-\infty} e^u = 0$, we have

$$\lim_{n\to\infty} |x^n| = \lim_{n\to\infty} |x|^n = \lim_{n\to\infty} e^{n \ln |x|} = \lim_{u\to-\infty} e^u = 0.$$

Therefore,

$$\lim_{n\to\infty} x^n = 0 \quad \text{if } |x| < 1.$$

(The sequence x^n, $|x| > 1$, is unbounded, thus divergent.)

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Since $-|x^n| \le x^n \le |x^n|$, by the pinching theorem, we have

$$\lim_{n\to\infty} x^n = 0 \quad \text{if } |x| < 1.$$

(The sequence x^n, $|x| > 1$, is unbounded, thus divergent.)

Alembert's Rule
Alembert's Rule

Let $(a_n)_{n=1}^{\infty}$, $a_n \neq 0\ \forall n$ such that $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \rho$. If $\rho < 1$, then the sequence $(a_n)_{n=1}^{\infty}$ converges to 0, and, if $\rho > 1$, it diverges.

Proof. $(\rho < 1)$
Since $0 < \dfrac{1+\rho}{2} < 1$, $\lim\limits_{n\to\infty} \left(\dfrac{1+\rho}{2} \right)^n = 0$. Since $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \rho$, $\exists N_1$ s.t. $\forall n > N_1$, $\left| \dfrac{a_{n+1}}{a_n} \right| \le \rho + \dfrac{1-\rho}{2}$, thus $|a_{n+1}| \le \dfrac{1+\rho}{2} |a_n|$. Then, for $n > N_1$,

$$|a_n| \leq \frac{1+\rho}{2}|a_{n-1}| \leq \left(\frac{1+\rho}{2}\right)^2 |a_{n-2}| \leq \cdots \leq \left(\frac{1+\rho}{2}\right)^{n-N_1} |a_{N_1}|$$

and therefore $\lim_{n\to\infty} |a_n| = 0$. Since $-|a_n| \leq a_n \leq |a_n|$, by the pinching theorem, we have $\lim_{n\to\infty} a_n = 0$.

Proof. $(\rho > 1)$
$\exists \delta > 0$ s.t. $\rho > 1+\delta$. Since $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$, $\exists N_2$ s.t. $\forall n > N_2$, $\left|\frac{a_{n+1}}{a_n}\right| \geq \rho - \frac{\delta}{2}$.
As $\rho - \frac{\delta}{2} > 1 + \frac{\delta}{2}$, $|a_{n+1}| \geq \left(1+\frac{\delta}{2}\right)|a_n|$. Then, for $n > N_1$,

$$|a_n| \geq \left(1+\frac{\delta}{2}\right)|a_{n-1}| \geq \left(1+\frac{\delta}{2}\right)^2 |a_{n-2}| \geq \cdots \geq \left(1+\frac{\delta}{2}\right)^{n-N_2} |a_{N_2}|$$

and therefore (a_n) is unbounded, thus divergent.

Some Important Limits: 4

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Proof.
Note that
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)!}\frac{n!}{|x|^n} = \frac{|x|}{n+1} \to 0 \quad \text{as } n \to \infty$$
By Alembert's Rule,
$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Some Important Limits: 5

$$\lim_{n\to\infty} \frac{n^p}{x^n} = 0, \quad |x| > 1.$$

Proof.
Note that
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^p}{|x|^{n+1}}\frac{|x|^n}{n^p} = \frac{1}{|x|}\left(1+\frac{1}{n}\right) \to \frac{1}{|x|} \quad \text{as } n \to \infty$$
Since $\frac{1}{|x|} < 1$, by Alembert's Rule,
$$\lim_{n\to\infty} \frac{n^p}{x^n} = 0, \quad |x| > 1.$$

Some Important Limits: 6

$$\lim_{n\to\infty}\frac{\ln n}{n}=0.$$

Proof.

$$0\le\frac{\ln n}{n}=\frac{1}{n}\int_1^n\frac{dt}{t}\le\frac{1}{n}\int_1^n\frac{dt}{\sqrt{t}}=\frac{2}{\sqrt{n}}(\sqrt{n}-1)\to 0 \text{ as } n\to 0.$$

By the pinching theorem, $\lim_{n\to\infty}\frac{\ln n}{n}=0$. Alternative proof is done by L'Hôpital's Rule (10.5),

$$\lim_{n\to\infty}\frac{\ln n}{n}=\lim_{u\to\infty}\frac{\ln u}{u}=\lim_{u\to\infty}\frac{\frac{1}{u}}{1}=0.$$

$$\lim_{n\to\infty}n^{\frac{1}{n}}=1.$$

Proof.

$$\lim_{n\to\infty}n^{\frac{1}{n}}=\lim_{n\to\infty}e^{\frac{\ln n}{n}}=\lim_{u\to 0}e^u=e^0=1.$$

Some Important Limits: 7

$$\lim_{n\to\infty}\frac{\ln n}{n^\alpha}=0,\quad \alpha>0.$$

Proof.
By L'Hôpital's Rule (10.5),

$$\lim_{n\to\infty}\frac{\ln n}{n^\alpha}=\lim_{u\to\infty}\frac{\ln u}{u^\alpha}=\lim_{u\to\infty}\frac{u^{-1}}{\alpha u^{\alpha-1}}=\lim_{u\to\infty}\frac{1}{\alpha u^\alpha}=0.$$

Some Important Limits: 8

$$\lim_{n\to\infty}n\ln\left(1+\frac{x}{n}\right)=x.$$

Proof.

$$\begin{aligned}\lim_{n\to\infty}n\ln\left(1+\frac{x}{n}\right)&=x\lim_{n\to\infty}\left[\frac{\ln\left(1+\frac{x}{n}\right)-\ln 1}{\frac{x}{n}}\right]\\&=x\lim_{h\to 0}\left[\frac{\ln(1+h)-\ln 1}{h}\right]\\&=x\lim_{h\to 0}\left[\frac{\ln(u+h)-\ln u}{h}\right]_{u=1}\\&=x(\ln u)'_{u=1}=x\frac{1}{u}|_{u=1}=x.\end{aligned}$$

Alternative proof is done by L'Hôpital's Rule (10.5),

$$\lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right) = \lim_{u\to\infty} \frac{\ln(1+x/u)}{1/u} = \lim_{u\to\infty} \frac{1/(1+x/u)(-x/u^2)}{-1/u^2} = x.$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

Proof.

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{x}{n}\right)} = \lim_{u\to x} e^u = e^x.$$

Outline

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