Lecture 20 Section 10.7 Improper Integrals

Jiwen He

1 Improper Integrals

What are Improper Integrals?

$$\int_1^\infty \frac{1}{x^2}\,dx = ?, \qquad \int_0^1 \frac{1}{x^2}\,dx = ?$$

Known: $\displaystyle\int_a^b f(x)\,dx = \int_a^b \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_a^b = \frac{1}{a} - \frac{1}{b}, \quad 0 < a < b,$

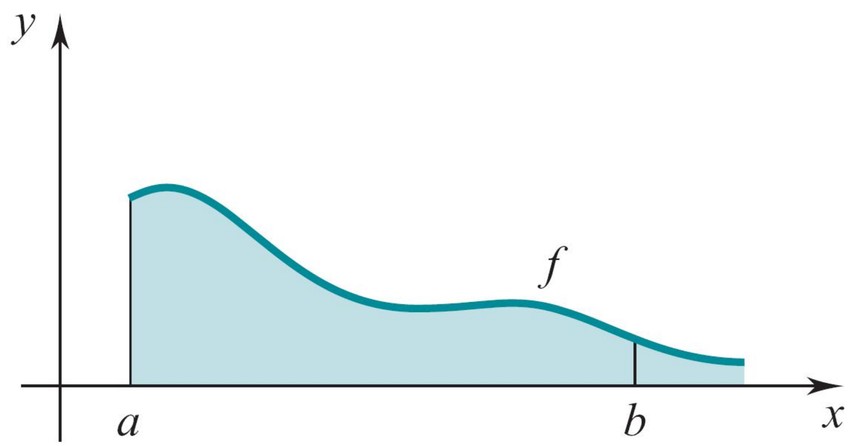
- the *interval* of integration $[a, b]$, $0 < a < b$, is *bounded*,
- the *function* being integrated $f(x) = \frac{1}{x^2}$ is *bounded* over $[a, b]$.

By a limit process, we can extend the integration process to

- *unbounded intervals* (e.g., $[1, \infty)$): $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(\frac{1}{1} - \frac{1}{b}\right) = 1$
- *unbounded functions* (e.g., as $x \to 0^+$, $f(x) = \frac{1}{x^2} \to \infty$): $\displaystyle\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+} \left(\frac{1}{a} - \frac{1}{1}\right) = \infty$

1.1 Integrals Over Unbounded Intervals

Integrals Over Unbounded Intervals



Let f be *continuous* on $[a, \infty)$. We define[0.5ex] $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

The improper integral *converges* if the limit exists. The improper integral *diverges* if the limit doesn't exist.

If $\lim_{b\to\infty} f(x) \neq 0$, then $\int_a^\infty f(x)\,dx$ *diverges.*

If f is *continuous* on $(-\infty, b]$, $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$. If f is *cont.* on $(-\infty, \infty)$, $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$

Fundamental Theorem of Integral Calculus
Let $f(x) = F'(x)$ for $\forall x \geq a$. Then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx = \lim_{b\to\infty}\left(F(x)\big|_a^b\right) = \lim_{b\to\infty} F(b) - F(a) = F(x)\big|_a^\infty = \lim_{x\to\infty} F(x) - F(a)$$

Let $f(x) = F'(x)$ for $\forall x \leq b$. Then

$$\int_{-\infty}^b f(x)\,dx = F(x)\big|_{-\infty}^b = F(b) - \lim_{x\to-\infty} F(x)$$

Let $f(x) = F'(x)$ for $\forall x$. Then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx = F(x)\big|_{-\infty}^\infty = \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x)$$
$$= \left(F(0) - \lim_{x\to-\infty} F(x)\right) + \left(\lim_{x\to\infty} F(x) - F(0)\right)$$

Examples

$$\int_1^\infty \frac{1}{x^3}\,dx = -\frac{2}{x^2}\bigg|_1^\infty = \lim_{x\to\infty}\left(-\frac{2}{x^2}\right) - (-2) = 2$$

$$\int_1^\infty \frac{1}{x}\,dx = \ln x|_1^\infty = \lim_{x\to\infty}(\ln x) - \ln 1 = \infty$$

$$\int_0^\infty e^{-2x}\,dx = -\frac{e^{-2x}}{2}\bigg|_0^\infty = \lim_{x\to\infty}\left(-\frac{e^{-2x}}{2}\right) - \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\begin{aligned}\int_{-\infty}^\infty \frac{c}{c^2+x^2}\,dx &= \tan^{-1}\left(\frac{x}{c}\right)\bigg|_{-\infty}^\infty \\ &= \lim_{x\to\infty}\left(\tan^{-1}\left(\frac{x}{c}\right)\right) - \lim_{x\to-\infty}\left(\tan^{-1}\left(\frac{x}{c}\right)\right) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi\end{aligned}$$

What is Wrong with This?
What is *wrong* with the following argument?

$$\infty = \int_{-\infty}^\infty \frac{2x}{1+x^2}\,dx = \lim_{b\to\infty}\int_{-b}^b \frac{2x}{1+x^2}\,dx = \lim_{b\to\infty}\left(\ln(1+x^2)\big|_{-b}^b\right) \neq \lim_{b\to\infty}\int_{-b}^b \frac{2x}{1+x^2}\,dx = 0$$

$$\int_0^\infty \frac{2x}{1+x^2}\,dx = \lim_{b\to\infty}\left(\ln(1+b^2) - \ln(1+(-b)^2)\right) = \lim_{b\to\infty}(0) = 0 = \ln(1+x^2)\big|_0^\infty = \lim_{x\to\infty}\ln(1+x^2) - \ln 1$$

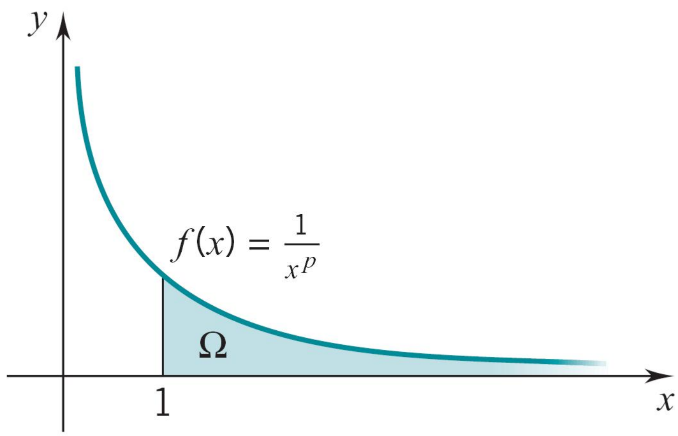
- Note that we *did not define* $\int_{-\infty}^\infty f(x)\,dx$ as $\lim_{b\to\infty}\int_{-b}^b f(x)\,dx$.

$$\int_{-\infty}^\infty f(x)\,dx = L \quad\Rightarrow\quad \lim_{b\to\infty}\int_{-b}^b f(x)\,dx = L$$

$$\lim_{b\to\infty}\int_{-b}^b f(x)\,dx = L \quad\not\Rightarrow\quad \int_{-\infty}^\infty f(x)\,dx = L$$

- For every odd function f, we have $\lim_{b\to\infty}\int_{-b}^b f(x)\,dx = 0$; but this is certainly not the case for $\int_{-\infty}^\infty f(x)\,dx$.

$\int_1^\infty \frac{1}{x^\alpha}\,dx$**,** $\alpha > 0$



$$\int_1^\infty \frac{1}{x^\alpha}\,dx = \begin{cases} \dfrac{1}{\alpha-1}, & \text{if } \alpha > 1, \\ \infty, & \text{if } \alpha \le 1. \end{cases}$$

If $\alpha \neq 1$, then

$$\int_1^\infty \frac{1}{x^\alpha}\,dx = \frac{1}{1-\alpha}x^{1-\alpha}\Big|_1^\infty = \frac{1}{1-\alpha}\lim_{x\to\infty} x^{1-\alpha} - \frac{1}{1-\alpha}$$
$$= \begin{cases} \frac{1}{\alpha-1}, & \text{if } \alpha > 1, \\ \infty, & \text{if } \alpha < 1. \end{cases}$$

If $\alpha = 1$, then

$$\int_1^\infty \frac{1}{x}\,dx = \ln x|_1^\infty = \lim_{x\to\infty}(\ln x) - \ln 1 = \infty$$

Quiz

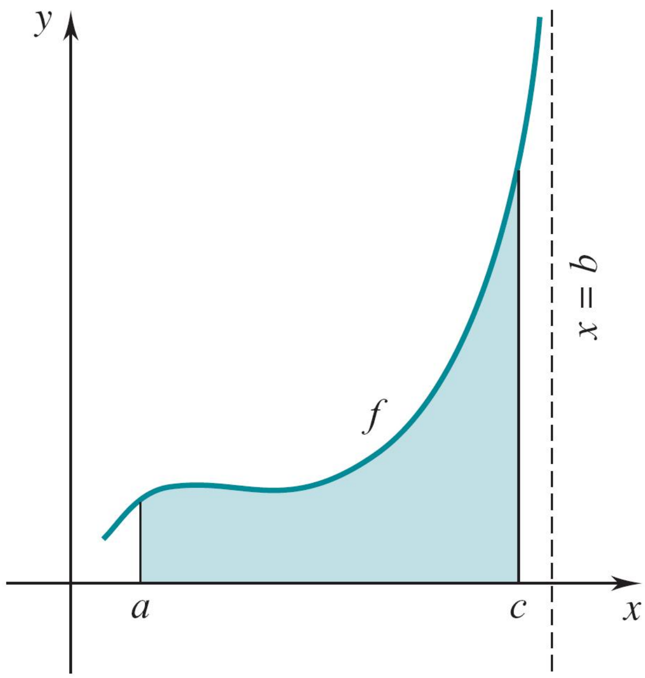
Quiz

1. Find $\lim_{x\to\infty} \frac{x^2}{e^x}$
   (a) 0, (b) 1, (c) ∞.

2. Find $\lim_{x\to\infty} \frac{x}{\ln x}$
   (a) 0, (b) 1, (c) ∞.

1.2 Integrals of Unbounded Functions

Integrals of Unbounded Functions



Let f be *continuous* on $[a, b)$, but $f(x) \to \pm\infty$ as $x \to b^-$. We define

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

The improper integral *converges* if the limit exists. The improper integral *diverges* if the limit doesn't exist.

If f is *continuous* on $(a, b]$, but $f(x) \to \pm\infty$ as $x \to a^+$.,

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

If f is *cont.* on $[a, b]$, except at *some point* c in (a, b) where $f(x) \to \pm\infty$ *as* $x \to c^-$ *or* $x \to c^+$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Fundamental Theorem of Integral Calculus

Let $f(x) = F'(x)$ for $\forall x \in [a, b)$ and $f(x) \to \pm\infty$ as $x \to b^-$. Then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx = \lim_{c \to b^-} \left(F(x)\big|_a^c\right) = \lim_{c \to b^-} F(c) - F(a) = F(x)\Big|_a^b = \lim_{x \to b^-} F(x) - F(a)$$

If $\lim_{x \to b^-} F(x) = \pm\infty$, then $\int_a^b f(x)\,dx$ *diverges.*

Let $f(x) = F'(x)$ for $\forall x \in (a, b]$ and $f(x) \to \pm\infty$ as $x \to a^+$. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - \lim_{x \to a^+} F(x)$$

Examples

$$\int_0^1 \frac{1}{x^{1/3}}\,dx = \frac{3}{2}x^{2/3}\Big|_0^1 = \frac{3}{2} - \frac{3}{2}\lim_{x\to 0^+}\left(x^{2/3}\right) = \frac{3}{2}$$

$$\int_0^1 \frac{1}{x}\,dx = \ln x|_0^1 = \ln 1 - \lim_{x\to 0^+}(\ln x) = \infty$$

$$\int_0^{\frac{\pi}{2}} \tan x\,dx = \ln\sec x|_0^{\frac{\pi}{2}} = \lim_{x\to\frac{\pi}{2}^-}(\ln\sec x) - \ln 1 = \infty$$

Examples

$$\int_0^\infty \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx = \int_1^0 \frac{-du}{\sqrt{1-u^2}} = \int_0^1 \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1} u\big|_0^1 = \frac{\pi}{2}$$

Set $u = e^{-x}$, then $du = -e^{-x}\,dx$ and $u^2 = e^{-2x}$.

$$\int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{2\cos x - 1}}\,dx = \int_1^0 \frac{-\frac{1}{2}du}{\sqrt{u}} = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{u}}\,du$$

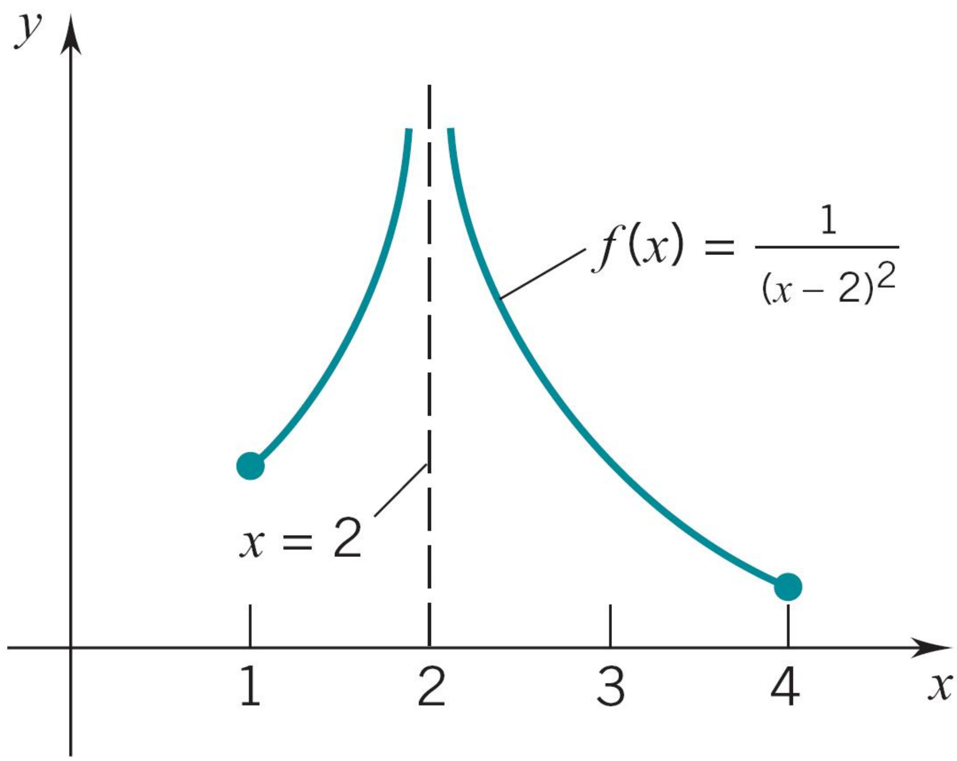
$$= \frac{1}{4}\,\sqrt{u}\big|_0^1 = \frac{1}{4}$$

Set $u = 2\cos x - 1$, then $du = -2\sin x\,dx$.

What is Wrong with This?
What is *wrong* with the following argument?

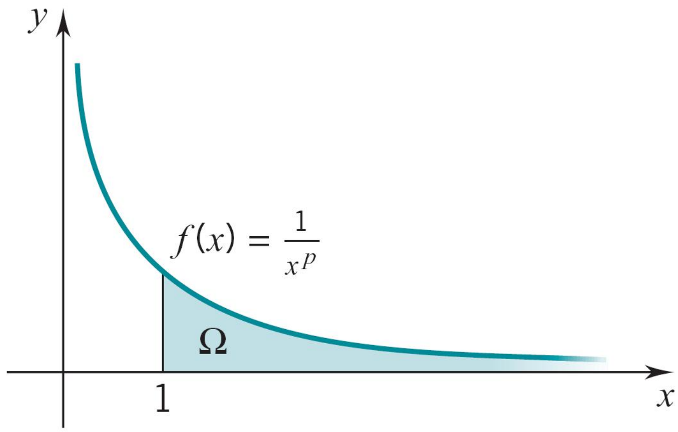
$$\infty = \int_1^4 \frac{1}{(x-2)^2}\,dx = \neq -\frac{1}{x-2}\Big|_1^4 = -\frac{3}{2}$$

$$\int_1^2 \frac{1}{(x-2)^2}\,dx = -\frac{1}{x-2}\Big|_1^2 = \lim_{x\to 2^-}\left(-\frac{1}{x-2}\right) - 1 = \infty$$



Note that $\frac{1}{(x-2)^2} \to \infty$ as $x \to 2^-$ or $x \to 2^+$. To evaluate $\displaystyle\int_1^4 \frac{1}{(x-2)^2}\,dx$ we need to calculate the *improper integrals* $\displaystyle\int_1^2 \frac{1}{(x-2)^2}\,dx$ and $\displaystyle\int_2^4 \frac{1}{(x-2)^2}\,dx$.

$\int_0^1 \frac{1}{x^\alpha}\,dx,\ \alpha > 0$



$$\int_0^1 \frac{1}{x^\alpha}\,dx = \begin{cases} \dfrac{1}{1-\alpha}, & \text{if } \alpha < 1, \\ \infty, & \text{if } \alpha \geq 1. \end{cases}$$

If $\alpha \neq 1$, then

$$\int_0^1 \frac{1}{x^\alpha}\,dx = \frac{1}{1-\alpha}x^{1-\alpha}\Big|_0^1 = \frac{1}{1-\alpha} - \frac{1}{1-\alpha}\lim_{x\to 0^+} x^{1-\alpha}$$
$$= \begin{cases} \frac{1}{1-\alpha}, & \text{if } \alpha < 1, \\ \infty, & \text{if } \alpha > 1. \end{cases}$$

If $\alpha = 1$, then

$$\int_0^1 \frac{1}{x}\,dx = \ln x|_0^1 = \ln 1 - \lim_{x\to 0^+}(\ln x) = \infty$$

Quiz

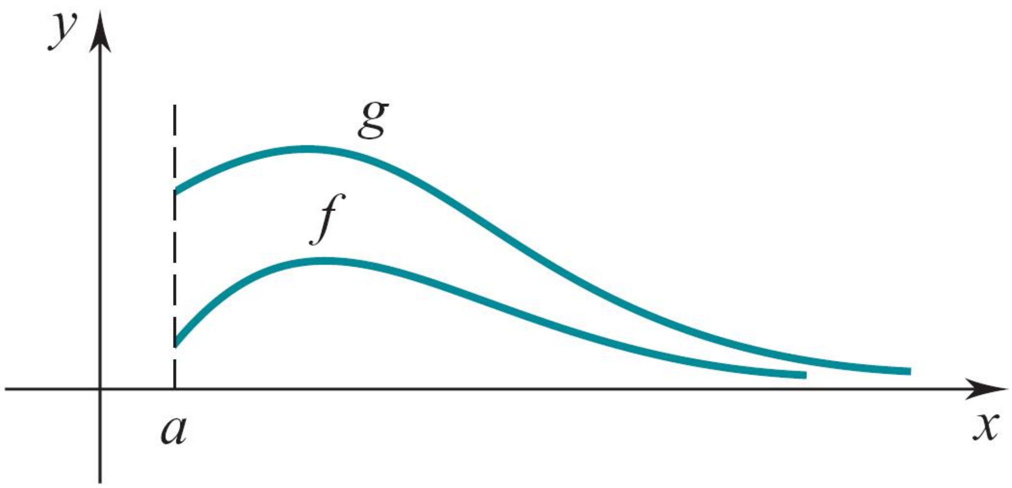
Quiz

3. Find $\lim_{x\to\infty} x \sin\frac{1}{x}$
   (a) 0, (b) 1, (c) ∞.

4. Find $\lim_{x\to 0^+} \frac{x}{e^x}$
   (a) 0, (b) 1, (c) ∞.

2 Comparison Test for Convergence

2.1 Comparison Test for Convergence



Let $\int_I$ be *improper* integration over an interval I, and suppose that f and g are continuous on I such that

$$0 \le f(x) \le g(x), \quad \forall x \in I$$

- If $\int_I g(x)\,dx$ *converges*, then so does $\int_I f(x)\,dx$.
- If $\int_I f(x)\,dx$ *diverges*, then so does $\int_I g(x)\,dx$.

The improper integral $\int_1^\infty \frac{1}{\sqrt{1+x^3}}\,dx$ *converges* since

$$0 \le \frac{1}{\sqrt{1+x^3}} < \frac{1}{\sqrt{x^3}}, \ \forall x \ge 1 \quad \text{and} \quad \int_1^\infty \frac{1}{\sqrt{x^3}}\,dx \ \text{ converges.}$$

The improper integral $\int_1^\infty \frac{1}{\sqrt{1+x^2}}\,dx$ *diverges* since

$$0 \le \frac{1}{1+x} < \frac{1}{\sqrt{1+x^2}}, \ \forall x \ge 1 \quad \text{and} \quad \int_1^\infty \frac{1}{1+x}\,dx \ \text{ diverges.}$$

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