Lecture 21 Section 11.1 Infinite Series

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1 Infinite Series

What is an Infinite Series?
Let the *sequence* $a_k = \frac{1}{2^k}$: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$. Form the *partial sums*

$$
\begin{aligned}
s_1 &= a_1 = \sum_{k=1}^{1} a_k = \frac{1}{2} \\
s_2 &= a_1 + a_2 = \sum_{k=1}^{2} a_k = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
s_3 &= a_1 + a_2 + a_3 = \sum_{k=1}^{3} a_k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\
s_4 &= a_1 + a_2 + a_3 + a_4 = \sum_{k=1}^{4} a_k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} \\
&\vdots \\
s_n &= \sum_{k=1}^{n} a_k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + + \frac{1}{16} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^{n+1}} \\
&\vdots \\
s_\infty &= \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \left(1 - \frac{1}{2^{n+1}}\right) = 1
\end{aligned}
$$

The "infinite sum" $\sum_{k=1}^{\infty} a_k$, is called *infinite series*.

Quiz

Quiz

1. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx =$

   (a) 1, (b) 2, (c) ∞.

2. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx =$

   (a) 1, (b) 2, (c) ∞.

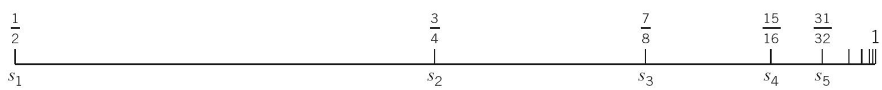
1.1 Basic Properties

Infinite Series: Definition

Given a *sequence* $\{a_k\}_{k=1}^{\infty}$, the *infinite series* is defined by the *limit of the sequence of partial sums*:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

The series *converges* if the limit exists; otherwise, it *diverges*.



$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = 1$$

Remarks
The *summation index* is a "*dummy*" index, much like the *integration variable* in a integral

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{i=1}^{\infty} \frac{1}{2^i} = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

The *starting value* of the summation index needn't be 1 and may be chosen for convenience

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = \sum_{n=2}^{\infty} \frac{1}{2^{n-1}}$$

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right) = 1$$

$$s_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{i} - \frac{1}{i+1} = 1 - \frac{1}{i+1} \to 1 \text{ as } n \to \infty$$

Basic Test for Convergence

$$\text{If } \sum_{k=1}^{\infty} a_k \text{ converges, then } a_k \to 0 \text{ as } k \to \infty$$

$$\text{If } a_k \nrightarrow 0 \text{ as } k \to \infty, \text{ then } \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

Example 1.

$$\frac{k}{k+1} \to 1 \neq 0 \text{ as } k \to \infty, \text{ then } \sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \cdots \text{ diverges.}$$

Remark
There are divergent series for which $a_k \to 0$:

$$\frac{1}{k} \to 0 \text{ as } k \to \infty, \text{ but } \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \text{ diverges.}$$

Basic Properties

- If $\sum_{k=1}^{\infty} a_k$ converges and c is a constant, then $\sum_{k=1}^{\infty} c\, a_k$ converges, and
$$\sum_{k=1}^{\infty} c\, a_k = c \sum_{k=1}^{\infty} a_k.$$
- If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, then $\sum_{k=1}^{\infty}(a_k + b_k)$ converges, and
$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

$$\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{\infty} a_k + \beta \sum_{k=1}^{\infty} b_k, \quad \forall \alpha, \beta \in \mathbb{R}$$

Quiz

Quiz

3. $\displaystyle\int_0^1 \frac{1}{x}\, dx =$

   (a) 1, (b) 2, (c) ∞.

4. $\displaystyle\int_1^{\infty} \frac{1}{x}\, dx =$

   (a) 1, (b) 2, (c) ∞.

1.2 Geometric Series

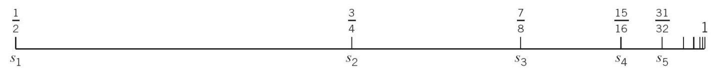
Geometric Series
Geometric Series: $\sum_{k=0}^{\infty} x^k$

$$\sum_{k=0}^{\infty} x^k = \begin{cases} \dfrac{1}{1-x}, & \text{if } |x| < 1, \\ \text{diverges}, & \text{if } |x| \geq 1. \end{cases}$$

Proof.

- We have a "closed" formula for the nth partial sum:
$$s_n = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}, \quad \text{if } x \neq 1$$
- If $|x| < 1$, then $x^{n+1} \to 0$ as $n \to \infty$, thus $s_n \to \frac{1}{1-x}$.
- If $|x| > 1$, then x^{n+1} is unbounded, thus s_n diverges.
- If $x = 1$, then $s_n = n + 1 \to \infty$ as $n \to \infty$
- If $x = -1$, then s_n alternates between 0 and 1, thus diverges.

Examples



$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{k=0}^{\infty} \frac{1}{2^k} - 1 = \frac{1}{1-\frac{1}{2}} - 1 = 1$$

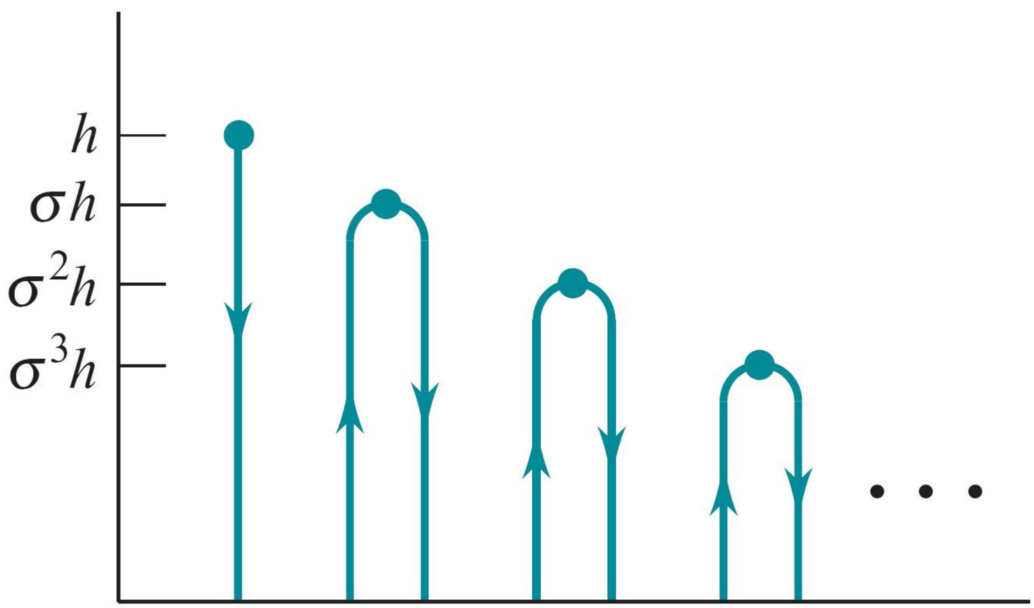
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$$

$$\sum_{k=1}^{\infty} \frac{1}{10^k} = 0.1111111\cdots = \sum_{k=0}^{\infty} \frac{1}{10^k} - 1 = \frac{1}{1-\frac{1}{10}} - 1 = \frac{1}{9}$$

$$\sum_{k=1}^{\infty} \frac{9}{10^k} = 0.9999999\cdots = 9\sum_{k=1}^{\infty} \frac{1}{10^k} = 9 \times \frac{1}{9} = 1$$

$$\Rightarrow a_k \in \{0, \cdots, 9\},\ \sum_{k=1}^{\infty} \frac{a_k}{10^k} = 0.a_1a_2a_3a_4a_5\cdots = \text{infinite decimal}$$

Examples



Suppose that a ball dropped from a height h hits the floor and rebounds to a height σh with $\sigma < 1$, and so on. Find the total distance traveled by the ball if h is 6 feet and $\sigma = \frac{2}{3}$.

The total distance traveled by the ball is

$$\begin{aligned} D &= h + 2\sigma h + 2\sigma^2 h + 2\sigma^3 h + \cdots \\ &= h + 2\sigma h \left(1 + \sigma + \sigma^2 + \cdots\right) = h + 2\sigma h \sum_{k=0}^{\infty} \sigma^k \\ &= h + 2\sigma h \frac{1}{1-\sigma} = 6 + 2 \times \frac{2}{3} \times 6 \frac{1}{1-\frac{2}{3}} = 36 \text{ feet} \end{aligned}$$

Examples

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{3^{k+1} - 2^k}{5^k} &= 2 + \frac{7}{5} + \frac{23}{25} + \frac{73}{125} + \cdots = \sum_{k=0}^{\infty} \frac{3^{k+1}}{5^k} - \sum_{k=0}^{\infty} \frac{2^k}{5^k} \\ &= 3 \sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k - \sum_{k=0}^{\infty} \left(\frac{2}{5}\right)^k = 3 \frac{1}{1-\frac{3}{5}} - \frac{1}{1-\frac{2}{5}} = \frac{35}{6} \\ \sum_{k=3}^{\infty} \frac{1}{3^k} &= \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots = \sum_{k=0}^{\infty} \frac{1}{3^k} - \left(1 + \frac{1}{3} + \frac{1}{9}\right) \\ &= \frac{1}{1-\frac{1}{3}} - \frac{13}{9} = \frac{1}{18} \\ \sum_{k=3}^{\infty} \frac{1}{3^k} &= \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots = \sum_{i=0}^{\infty} \frac{1}{3^{i+3}} = \frac{1}{27} \sum_{k=0}^{\infty} \frac{1}{3^k} \\ &= \frac{1}{27} \frac{1}{1-\frac{1}{3}} = \frac{1}{18} \end{aligned}$$

Outline

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