Lecture 22 Section 11.2 The Integral Test; Comparison Tests

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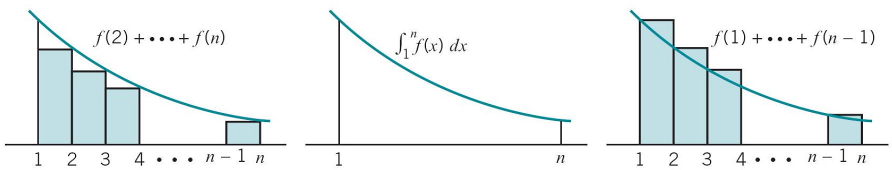
1 The Integral Test

1.1 The Integral Test

The Integral Test



Let $a_k = f(k)$, where f is continuous, *decreasing* and *positive* on $[1, \infty)$, then

$$\sum_{k=1}^{\infty} a_k \text{ converges} \quad \textit{iff} \quad \int_1^{\infty} f(x)dx \text{ converges}$$

Since f decreases on $[1, \infty)$ $\displaystyle\sum_{k=2}^{n} a_k < \int_1^n f(x)dx < \sum_{k=1}^{n-1} a_k$. Since f is positive on $[1, \infty)$, $\displaystyle\sum_{k=2}^{\infty} a_k \leq \int_1^{\infty} f(x)dx \leq \sum_{k=1}^{\infty} a_k$. Therefore, either both converge or both diverge.

1.2 Applying the Integral Test

The p-Sereis
(The p-Sereis)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \text{ converges} \quad \textit{iff} \quad p > 1.$$

Proof.

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges} \quad \textit{iff} \quad \int_1^{\infty} \frac{1}{x^p} dx \text{ converges.}$$

We know that

$$\int_1^{\infty} \frac{1}{x^p} dx \text{ converges} \quad \textit{iff} \quad p > 1.$$

It follows that

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges} \quad \textit{iff} \quad p > 1.$$

Example $(p = 1)$**: Harmonic Series**
(The p**-Sereis)**

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \text{ converges } \quad \textit{iff} \quad p > 1.$$

Harmonic Series $(p = 1)$
When $p = 1$, this is the *harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots,$$

and the related improper integral is

$$\int_1^{\infty} \frac{1}{x}\, dx = \ln x\Big|_1^{\infty} = \lim_{x\to\infty} \ln x = \infty.$$

Since this impproper integral *diverges*, so does the harmonic series.

Example $(p = 2)$
(The p**-Sereis)**

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \text{ converges } \quad \textit{iff} \quad p > 1.$$

$(p = 2)$
When $p = 2$, this is
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots,$$
and the related improper integral is
$$\int_1^{\infty} \frac{1}{x^2}\, dx = -x^{-1}\Big|_1^{\infty} = 1 - \lim_{x\to\infty} x^{-1} = 1.$$
By the integral test,
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \sum_{k=2}^{\infty} \frac{1}{k^2} \le 1 + \int_1^{\infty} \frac{1}{x^2}\, dx = 2$$

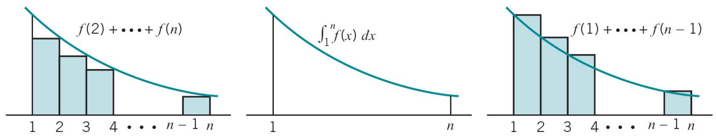
Example
Show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges.

The related improper integral is
$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u}\, du = \ln u\Big|_{\ln 2}^{\infty} = \lim_{x\to\infty} \ln x - \ln\ln 2 = \infty.$$
Since this impproper integral *diverges*, so does the infinite series.

Show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges.

The related improper integral is
$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u^2}\, du = -u^{-1}\Big|_{\ln 2}^{\infty} = \frac{1}{\ln 2} - \lim_{x\to\infty} x^{-1} = \frac{1}{\ln 2}.$$
Since this impproper integral *diverges*, so does the infinite series.

The Integral Test for the nth Remainder $\sum_{k=n+1}^{\infty} a_k$, $n \geq 1$



Let $a_k = f(k)$, where f is continuous, *decreasing* and *positive* on $[n, \infty)$, then

$$\sum_{k=n+1}^{\infty} a_k \leq \int_n^{\infty} f(x)dx \leq a_n + \sum_{k=n+1}^{\infty} a_k$$

or equivalently

$$\int_{n+1}^{\infty} f(x)dx \leq \sum_{k=n+1}^{\infty} a_k \leq \int_n^{\infty} f(x)dx$$

Let $\sum_{k=1}^{\infty} a_k$ be convergent. Its *nth remainder* is defined as

$$R_n = \sum_{k=n+1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k - s_n$$

Let $\sum_{k=1}^{\infty} a_k$ be convergent, so do $\int_1^{\infty} f(x)dx$. Then, $\forall \epsilon > 0$, $\exists N$ s.t. for $n > N$, $\int_n^{\infty} f(x)dx < \epsilon$, and

$$0 < R_n = \sum_{k=n+1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k < \epsilon$$

Example

Use a partial sum to approximate $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ with an error < 0.01

Estimating the nth remainder of the series

$$R_n = \sum_{k=1}^{\infty} \frac{1}{k^2+1} - \sum_{k=1}^{n} \frac{1}{k^2+1} < \int_n^{\infty} \frac{1}{x^2+1}dx$$

$$= \tan^{-1} x\Big|_n^{\infty} = \frac{\pi}{2} - \tan^{-1} n.$$

For $\frac{\pi}{2} - \tan^{-1} n < 0.01$, we need $n \geq \tan(\frac{\pi}{2} - 0.01) \approx 100$.

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2+1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots \frac{1}{10001} = 1.066\ldots$$

Therefore

$$1.066\ldots < \sum_{k=1}^{\infty} \frac{1}{k^2+1} < 1.076\ldots$$

1.3 Comparison Tests

Basic Series that Converge or Diverge

$$\sum_{k=1}^{\infty} a_k \text{ converges} \quad \textit{iff} \quad \sum_{k=j}^{\infty} a_k \text{ converges}, \forall j \geq 1.$$

In determining whether a series converges, it does not matter where the summation begins. Thus, we will omit it and write $\sum a_k$.

Basic Series that Converge

$$\text{Geometric series:} \quad \sum x^k, \quad \text{if } |x| < 1$$

$$p\text{-series:} \quad \sum \frac{1}{k^p}, \quad \text{if } p > 1$$

Basic Series that Diverge

$$\text{Any series } \sum a_k \quad \text{for which } \lim_{k\to\infty} a_k \neq 0$$

$$p\text{-series:} \quad \sum \frac{1}{k^p}, \quad \text{if } p \leq 1$$

Basic Comparison Test

Basic Comparison Test

Suppose that $0 \leq a_k \leq b_k$ for sufficiently large k.

If $\sum b_k$ converges, then so does $\sum a_k$.

If $\sum a_k$ diverges, then so does $\sum b_k$.

Comparison with p-Series

$$\sum a_k \text{ converges if } \quad 0 \leq a_k \leq \frac{c}{k^p}, \quad p > 1.$$

$$\sum a_k \text{ diverges if } \quad 0 \leq \frac{c}{k^p} \leq a_k, \quad p \leq 1.$$

Comparison with Geometric Series

$$\sum a_k \text{ converges if } \quad 0 \leq a_k \leq c\,x^k, \quad |x| < 1.$$

Examples

$\sum \frac{1}{2k^3+1}$ converges by comparison with $\sum \frac{1}{k^3}$

$$\frac{1}{2k^3+1} < \frac{1}{k^3} \quad \text{and} \quad \sum \frac{1}{k^3} \text{ converges.}$$

$\sum \frac{k^3}{k^5+4k^4+7}$ converges by comparison with $\sum \frac{1}{k^2}$

$$\frac{k^3}{k^5+4k^4+7} < \frac{k^3}{k^5} = \frac{1}{k^2} \quad \text{and} \quad \sum \frac{1}{k^2} \text{ converges.}$$

$\sum \frac{1}{k^3-k^2}$ converges by comparison with $\sum \frac{2}{k^3}$

$$\frac{1}{k^3-k^2} < \frac{1}{k^3-k^3/2} = \frac{2}{k^3}, \ k \geq 2, \quad \text{and} \quad \sum \frac{2}{k^3} \text{ converges.}$$

Examples

$\sum \frac{1}{3k+1}$ diverges by comparison with $\sum \frac{1}{3(k+1)}$

$$\frac{1}{3k+1} > \frac{1}{3(k+1)} \quad \text{and} \quad \frac{1}{3}\sum \frac{1}{k+1} \textit{ diverges.}$$

$\sum \frac{1}{\ln(k+6)}$ diverges by comparison with $\sum \frac{1}{k+6}$

$$\frac{\ln(k+6)}{k+6} \to 0 \text{ as } k \to \infty, \quad \ln(k+6) < k+6 \text{ for } k \text{ large}$$

$$\frac{1}{\ln(k+6)} > \frac{1}{k+6} \text{ for } k \text{ large} \quad \text{and} \quad \sum \frac{1}{k+6} \textit{ diverges.}$$

Limit Comparison Test

Basic Comparison Test

Suppose that $a_k > 0$ and $b_k > 0$ for sufficiently large k, and that $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ for some $L > 0$.

$$\sum a_k \text{ converges} \quad \textit{iff} \quad \sum b_k \text{ converges.}$$

$\lim_{k\to\infty} \frac{a_k}{b_k} = L$ means that $a_k \approx L b_k$ for large k

If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$, and

$$\text{if } \sum b_k \text{ converges, then} \quad \sum a_k \text{ converges.}$$

If $\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$, and

$$\text{if } \sum a_k \text{ converges, then} \quad \sum b_k \text{ converges.}$$

Example

$\sum \frac{1}{k^3-1}$ converges by comparison with $\sum \frac{1}{k^3}$.

For large k, $\frac{1}{k^3-1}$ differs little from $\frac{1}{k^3}$.

$$\frac{1}{k^3-1} \div \frac{1}{k^3} = \frac{k^3}{k^3-1} \to 1 \text{ as } k \to \infty$$

and

$$\sum \frac{1}{k^3} \text{ converges.}$$

Example

$\sum \frac{3k^2+2k+1}{k^3+1}$ diverges by comparison with $\sum \frac{3}{k}$

For large k, $\frac{3k^2+2k+1}{k^3+1}$ differs little from $\frac{3k^2}{k^3} = \frac{3}{k}$.

$$\frac{3k^2+2k+1}{k^3+1} \div \frac{3}{k} = \frac{3k^3+2k^2+k}{3k^3+3} \to 1 \text{ as } k \to \infty$$

and

$$\sum \frac{3}{k} \text{ diverges.}$$

Example

$\sum \frac{5\sqrt{k}+100}{2k^2\sqrt{k}-9\sqrt{k}}$ converges by comparison with $\sum \frac{5}{2k^2}$

For large k, $\frac{5\sqrt{k}+100}{2k^2\sqrt{k}-9\sqrt{k}}$ differs little from $\frac{5\sqrt{k}}{2k^2\sqrt{k}} = \frac{5}{2k^2}$.

$$\frac{5\sqrt{k}+100}{2k^2\sqrt{k}-9\sqrt{k}} \div \frac{5}{2k^2} = \frac{10k^2\sqrt{k}+200k^2}{10k^2\sqrt{k}-45\sqrt{k}} \to 1 \text{ as } k \to \infty$$

and

$$\sum \frac{5}{2k^2} \text{ converges.}$$

Outline

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