## Lecture 23

# Section 11.3 The Root Test; The Ratio Test 

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## Basic Series that Converge or Diverge



In determining whether a series converges, it does not matter where the summation begins

Basic Series that Converge

Basic Series that Diverge

## Basic Series that Converge or Diverge

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\sum_{k=1}^{\infty} a_{k} \text { converges iff } \quad \sum_{k=j}^{\infty} a_{k} \text { converges, } \forall j \geq 1
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Geometric series: $\quad \sum x^{k}, \quad$ if $|x|<1$
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p \text {-series: } \quad \sum \frac{1}{k^{p}}, \quad \text { if } p>1
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Any series $\sum a_{k}$ for which $\lim _{k \rightarrow \infty} a_{k} \neq 0$

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$$

Basic Series that Diverge
Any series $\sum a_{k}$ for which $\lim _{k \rightarrow \infty} a_{k} \neq 0$

$$
p \text {-series: } \quad \sum \frac{1}{k^{p}}, \quad \text { if } p \leq 1
$$

## Quiz

## Quiz

$$
\begin{aligned}
& \text { 1. } \sum \frac{1}{n} \text { (a) converges, } \\
& \text { 2. } \sum \frac{1}{\sqrt{n}} \\
& \text { (b) diverges. } \\
& \text { (a) converges, }
\end{aligned} \text { (b) diverges. }
$$



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& \text { 1. } \sum \frac{1}{n} \\
& \text { (a) converges, } \\
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\end{aligned} \text { (b) diverges. } \text { (a) } \begin{aligned}
& \text { (anverges, } \\
& \text { (b) diverges. }
\end{aligned}
$$

1. $\sum \frac{1}{n}$ Harmonic series diverges.


## Quiz

## Quiz

1. $\sum \frac{1}{n}$ (a) converges, (b) diverges.
2. $\sum \frac{1}{\sqrt{n}}$
(a) converges, (b) diverges.
3. $\sum \frac{1}{n}$ Harmonic series diverges.
4. $\sum \frac{1}{\sqrt{n}} \quad p$-series with $p=\frac{1}{2}$ diverges.

## Comparison Tests

## Basic Comparison Test

Suppose that $0 \leq a_{k} \leq b_{k}$ for sufficiently large $k$.

## Limit Comparison Test

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Suppose that $0 \leq a_{k} \leq b_{k}$ for sufficiently large $k$.
If $\sum b_{k}$ converges, then so does $\sum a_{k}$.
 $a_{k}$ diverges, then so does


## Limit Comparison Test

Suppose that $a_{k}>0$ and $b_{k}>0$ for sufficiently large

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\end{aligned}
$$

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Suppose that $a_{k}>0$ and $b_{k}>0$ for sufficiently large $k$, and that $\lim _{k \rightarrow \infty} \frac{a_{k}}{b_{k}}=L$ for some $L>0$


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## Comparison Tests

## Basic Comparison Test

Suppose that $0 \leq a_{k} \leq b_{k}$ for sufficiently large $k$.

> If $\sum b_{k}$ converges, then so does $\sum a_{k}$.
> If $\sum a_{k}$ diverges, then so does $\sum b_{k}$.

## Limit Comparison Test

Suppose that $a_{k}>0$ and $b_{k}>0$ for sufficiently large $k$, and that $\lim _{k \rightarrow \infty} \frac{a_{k}}{b_{k}}=L$ for some $L>0$.

$$
\sum a_{k} \text { converges iff } \sum b_{k} \text { converges. }
$$

## Quiz

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$$
\text { 3. } \sum \frac{1}{n^{2}+1} \text { (a) converges, (b) diverges. }
$$

4. $\sum \frac{2 n}{\sqrt{3 n^{3}+5}}$
(a) converges, (b) diverges.



## Quiz

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$$
\begin{array}{ll}
\text { 3. } \sum \frac{1}{n^{2}+1} & \text { (a) converges, } \\
\text { 4. } \sum \frac{2 n}{\sqrt{3 n^{3}+5}} & \text { (a) diverges. } \\
\text { (anverges, } & \text { (b) diverges. }
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3. $\sum \frac{1}{n^{2}+1}$ converges by comparison with $\sum \frac{1}{n^{2}}$.


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3. $\sum \frac{1}{n^{2}+1}$ converges by comparison with $\sum \frac{1}{n^{2}}$.
4. $\sum \frac{2 n}{\sqrt{3 n^{3}+5}}$ converges by comparison with $\sum \frac{2}{3 n^{2}}$.

## The Root Test: Comparison with Geometric Series

## Root Test

Suppose that $a_{k}>0$ for large $k$, and that

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Suppose that $a_{k}>0$ for large $k$, and that

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\lim _{k \rightarrow \infty}\left(a_{k}\right)^{\frac{1}{k}}=\rho \text { for some } \rho>0
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- If $\rho<1$, then $\sum a_{k}$ converges.
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- If $\rho=1$, then the test is inconclusive.

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- If $\sum a_{k}$ is a geometric series, e.g., $\sum \rho^{k}, \rho>0$, then $\left(a_{k}\right)^{\frac{1}{k}}$ is constant, i.e., $\rho$. If $\rho<1$, then $\sum a_{k}$ converges. If $\rho \geq 1$, then $\sum a_{k}$ diverges.


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- If $\lim _{k \rightarrow \infty}\left(a_{k}\right)^{\frac{1}{k}}=\rho<1$, then for large $k, a_{k}<\mu^{k}$ with $\rho<\mu<1$. By the basic comparison test, $\sum a_{k}$ converges.


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## Examples



## by the root test:



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\left(a_{k}\right)^{1 / k}=\left(\frac{k^{2}}{2^{k}}\right)
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## Examples

$\sum_{2 x}^{k x^{2}}$
by the root test:

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## Examples

$\sum_{2 x}^{k}$
by the root test:
$\left(a_{k}\right)^{1 / k}=\left(\frac{k^{2}}{2^{k}}\right)^{1 / k}=\frac{1}{2} \cdot\left(k^{2}\right)^{1 / k}$

## Examples

$\sum_{2_{2}^{2}}^{\frac{k}{2}}$
by the root test:

$$
\left(a_{k}\right)^{1 / k}=\left(\frac{k^{2}}{2^{k}}\right)^{1 / k}=\frac{1}{2} \cdot\left(k^{2}\right)^{1 / k}
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## Examples

$\sum_{z z^{2}}^{k}$
by the root test:

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\begin{aligned}
\left(a_{k}\right)^{1 / k} & =\left(\frac{k^{2}}{2^{k}}\right)^{1 / k}=\frac{1}{2} \cdot\left(k^{2}\right)^{1 / k} \\
& =\frac{1}{2} \cdot\left[k^{1 / k}\right]^{2}
\end{aligned}
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$\sum \frac{k^{2}}{2^{k}}$ converges, by the root test:

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$\sum \frac{1}{(\ln k)^{k}}$
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$\sum \frac{1}{(\ln k)^{k}}$ by the root test:

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\begin{aligned}
\left(a_{k}\right)^{1 / k} & =\left(\frac{1}{(\ln k)^{k}}\right)^{1 / k} \\
& =\frac{1}{\ln k}
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## Examples

## $\sum\left(1-\frac{1}{k}\right)^{k^{2}}$ by the root test:


$\qquad$

## Examples

$$
\sum\left(1-\frac{1}{k}\right)^{k^{2}} \quad \text { by the root test: }
$$

$$
\left(a_{k}\right)^{1 / k}=\left(1-\frac{1}{k}\right)^{\prime}
$$

## Examples

$$
\sum\left(1-\frac{1}{k}\right)^{k^{2}} \quad \text { by the root test: }
$$



## Examples

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\sum\left(1-\frac{1}{k}\right)^{k^{2}} \quad \text { by the root test: }
$$

$$
\left(a_{k}\right)^{1 / k}=\left(1-\frac{1}{k}\right)^{k}=\left(1+\frac{(-1)}{k}\right)
$$

## Examples

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\begin{aligned}
& \sum\left(1-\frac{1}{k}\right)^{k^{2}} \\
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\end{aligned}
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\begin{aligned}
& \sum\left(1-\frac{1}{k}\right)^{k^{2}} \text { converges, by the root test: } \\
& \left(a_{k}\right)^{1 / k}=\left(1-\frac{1}{k}\right)^{k}=\left(1+\frac{(-1)}{k}\right)^{k} \rightarrow e^{-1}<1 \text { as } k \rightarrow \infty
\end{aligned}
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$$

$$
\sum\left(1-\frac{1}{k}\right)^{k} \quad \text { by the root test: }
$$

$$
\left(a_{k}\right)^{1 / k}=1-\frac{1}{k}
$$

## Examples

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\end{aligned}
$$

$$
\begin{aligned}
& \sum\left(1-\frac{1}{k}\right)^{k} \text { inconclusive, by the root test: } \\
& \qquad\left(a_{k}\right)^{1 / k}=1-\frac{1}{k} \rightarrow 1 \text { as } k \rightarrow \infty
\end{aligned}
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## Examples

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& \sum\left(1-\frac{1}{k}\right)^{k^{2}} \text { converges, by the root test: } \\
& \left(a_{k}\right)^{1 / k}=\left(1-\frac{1}{k}\right)^{k}=\left(1+\frac{(-1)}{k}\right)^{k} \rightarrow e^{-1}<1 \text { as } k \rightarrow \infty
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## Ratio Test

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- If $\lambda<1$, then $\sum a_{k}$ converges.
- If $\lambda>1$, then $\sum a_{k}$ diverges.
- If $\lambda=1$, then the test is inconclusive.


## Comparison with Geometric Series

$\square$
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## Comparison with Geometric Series

- If $\sum a_{k}$ is a multiple of a geometric series, e.g., $\sum c \lambda^{k}$,
$\lambda>0$, then $\frac{a_{k+1}}{a_{k}}$ is constant, i.e., $\lambda$. If $\lambda<1$, then $\sum a_{k}$ converges. If $\lambda \geq 1$, then $\sum a_{k}$ diverges.


## The Ratio Test: Comparison with Geometric Series

## Ratio Test

Suppose that $a_{k}>0$ for large $k$, and that

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\lim _{k \rightarrow \infty} \frac{a_{k+1}}{a_{k}}=\lambda \text { for some } \lambda>0
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- If $\lim _{k \rightarrow \infty} \frac{a_{k+1}}{a_{k}}=\lambda<1$, then for large $k, a_{k}<c \mu^{k}$ with $\lambda<\mu<1$. By the basic comparison test, $\sum a_{k}$ converges.


## Examples



## by the ratio test:

$$
\frac{a_{k}-1}{a_{k}}
$$

$\qquad$
$\square$

## Examples

$\sum \frac{k^{2}}{2^{k}}$
by the ratio test:


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## Examples

$\sum_{\frac{k}{z^{2}}}$
by the ratio test:

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\begin{aligned}
\frac{a_{k+1}}{a_{k}} & =\frac{(k+1)^{2}}{2^{k+1}} \div \frac{k^{2}}{2^{k}}=\frac{2^{k}}{2^{k+1}} \cdot \frac{(k+1)^{3}}{k^{3}} \\
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## Examples



## by the ratio test:

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\frac{a_{k+1}}{a_{k}}
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$\square$

## Examples

$\sum \frac{k}{10^{k}}$by the ratio test:


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$\sum \frac{k^{k}}{k!} \quad$ by the ratio test:

$$
\begin{equation*}
\frac{a_{k+1}}{a_{k}}=\frac{(k+1)^{k+1}}{(k+1)!} \div \frac{k^{k}}{k!} \tag{k+1}
\end{equation*}
$$

## Examples

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$\sum^{\frac{k^{k}}{k!}}$
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$$
\begin{aligned}
\frac{a_{k+1}}{a_{k}} & =\frac{(k+1)^{k+1}}{(k+1)!} \div \frac{k^{k}}{k!}=\frac{k!}{(k+1)!} \cdot \frac{(k+1)^{k+1}}{k^{k}}=\frac{(k+1)^{k}}{k^{k}} \\
& =\left(\frac{k+1}{k}\right)^{k}=\left(1+\frac{1}{k}\right)^{k}
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## Examples

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& =\left(\frac{k+1}{k}\right)^{k}=\left(1+\frac{1}{k}\right)^{k} \rightarrow e>1 \text { as } k \rightarrow \infty
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## Examples

$\sum \frac{k}{10^{k}}$ converges, by the ratio test:

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## Examples



## by the ratio test:

$\qquad$

## Examples

$\sum \frac{2^{k}}{3^{k}-2^{k}}$
by the ratio test:


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$$
\frac{a_{k+1}}{a_{k}}=\frac{2^{k+1}}{3^{k+1}-2^{k+1}} \div \frac{2^{k}}{3^{k}-2^{k}}
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## Examples

$$
\begin{aligned}
& \sum \frac{2^{k}}{3^{k}-2^{k}} \quad \text { by the ratio test: } \\
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& =2 \cdot \frac{1-(2 / 3)^{k}}{3-2(2 / 3)^{k}} \rightarrow 2 \cdot \frac{1}{3}<1 \text { as } k \rightarrow \infty
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## Outline

- Comparison Tests
- The Root Test
- The Root Test
- The Ratio Test
- The Ratio Test

