# Lecture 23 Section 11.3 The Root Test; The Ratio Test

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## 1 Comparison Tests

**Basic Series that Converge or Diverge**

$$\sum_{k=1}^{\infty} a_k \text{ converges} \quad \textit{iff} \quad \sum_{k=j}^{\infty} a_k \text{ converges}, \forall j \geq 1.$$

In determining whether a series converges, it does not matter where the summation begins. Thus, we will omit it and write $\sum a_k$.

**Basic Series that Converge**

$$\text{Geometric series:} \quad \sum x^k, \quad \text{if } |x| < 1$$

$$p\text{-series:} \quad \sum \frac{1}{k^p}, \quad \text{if } p > 1$$

**Basic Series that Diverge**

$$\text{Any series } \sum a_k \quad \text{for which } \lim_{k\to\infty} a_k \neq 0$$

$$p\text{-series:} \quad \sum \frac{1}{k^p}, \quad \text{if } p \leq 1$$

**Quiz**

**Quiz**

1. $\sum \frac{1}{n}$ (a) converges, (b) diverges.
2. $\sum \frac{1}{\sqrt{n}}$ (a) converges, (b) diverges.

1. $\sum \frac{1}{n}$ Harmonic series diverges.
2. $\sum \frac{1}{\sqrt{n}}$ $p$-series with $p = \frac{1}{2}$ diverges.

**Comparison Tests**

**Basic Comparison Test**
Suppose that $0 \leq a_k \leq b_k$ for sufficiently large $k$.

$$\text{If } \sum b_k \text{ converges, then so does } \sum a_k.$$
$$\text{If } \sum a_k \text{ diverges, then so does } \sum b_k.$$

**Limit Comparison Test**
Suppose that $a_k > 0$ and $b_k > 0$ for sufficiently large $k$, and that $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = L$ for some $L > 0$.

$$\sum a_k \text{ converges} \quad \textit{iff} \quad \sum b_k \text{ converges.}$$

**Quiz**

**Quiz**

3. $\sum \dfrac{1}{n^2+1}$ (a) converges, (b) diverges.

4. $\sum \dfrac{2n}{\sqrt{3n^3+5}}$ (a) converges, (b) diverges.

3. $\sum \dfrac{1}{n^2+1}$ converges by comparison with $\sum \dfrac{1}{n^2}$.

4. $\sum \dfrac{2n}{\sqrt{3n^3+5}}$ converges by comparison with $\sum \dfrac{2}{3n^2}$.

# 2 The Root Test

## 2.1 The Root Test

**The Root Test: Comparison with Geometric Series**
**Root Test**
Suppose that $a_k > 0$ for large $k$, and that [0.5ex] $\lim\limits_{k\to\infty} (a_k)^{\frac{1}{k}} = \rho$ for some $\rho > 0$.

- If $\rho < 1$, then $\sum a_k$ *converges.*
- If $\rho > 1$, then $\sum a_k$ *diverges.*
- If $\rho = 1$, then the test is *inconclusive.*

**Comparison with Geometric Series**

- If $\sum a_k$ is a geometric series, e.g., $\sum \rho^k$, $\rho > 0$, then $(a_k)^{\frac{1}{k}}$ is constant, i.e., $\rho$. If $\rho < 1$, then $\sum a_k$ converges. If $\rho \geq 1$, then $\sum a_k$ diverges.
- If $\lim\limits_{k\to\infty} (a_k)^{\frac{1}{k}} = \rho < 1$, then for large $k$, $a_k < \mu^k$ with $\rho < \mu < 1$. By the basic comparison test, $\sum a_k$ converges.

**Examples**

$\sum \dfrac{k^2}{2^k}$ *converges*, by the *root test*:

$$\begin{aligned}(a_k)^{1/k} &= \left(\frac{k^2}{2^k}\right)^{1/k} = \frac{1}{2}\cdot\left(k^2\right)^{1/k} \\ &= \frac{1}{2}\cdot\left[k^{1/k}\right]^2 \to \frac{1}{2}\cdot 1 < 1 \text{ as } k\to\infty\end{aligned}$$

$\sum \dfrac{1}{(\ln k)^k}$ *converges*, by the *root test*:

$$\begin{aligned}(a_k)^{1/k} &= \left(\frac{1}{(\ln k)^k}\right)^{1/k} \\ &= \frac{1}{\ln k} \to 0 \text{ as } k\to\infty\end{aligned}$$

**Examples**

$\sum \left(1-\dfrac{1}{k}\right)^{k^2}$ *converges*, by the *root test*:

$$(a_k)^{1/k} = \left(1-\frac{1}{k}\right)^k = \left(1+\frac{(-1)}{k}\right)^k \to e^{-1} < 1 \text{ as } k\to\infty$$

$\sum \left(1-\dfrac{1}{k}\right)^{k}$ *inconclusive*, by the *root test*:

$$(a_k)^{1/k} = 1-\frac{1}{k} \to 1 \text{ as } k\to\infty$$

the series *diverges* since $a_k = \left(1-\frac{1}{k}\right)^k \to e^{-1} \neq 0$.

# 3 The Ratio Test

## 3.1 The Ratio Test

**The Ratio Test: Comparison with Geometric Series**
**Ratio Test**
Suppose that $a_k > 0$ for large $k$, and that [0.5ex] $\lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k} = \lambda$ for some $\lambda > 0$.

- If $\lambda < 1$, then $\sum a_k$ *converges.*
- If $\lambda > 1$, then $\sum a_k$ *diverges.*
- If $\lambda = 1$, then the test is *inconclusive.*

**Comparison with Geometric Series**

- If $\sum a_k$ is a multiple of a geometric series, e.g., $\sum c\,\lambda^k$, $\lambda > 0$, then $\frac{a_{k+1}}{a_k}$ is constant, i.e., $\lambda$. If $\lambda < 1$, then $\sum a_k$ converges. If $\lambda \geq 1$, then $\sum a_k$ diverges.
- If $\lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k} = \lambda < 1$, then for large $k$, $a_k < c\mu^k$ with $\lambda < \mu < 1$. By the basic comparison test, $\sum a_k$ converges.

**Examples**

$\sum \frac{k^2}{2^k}$ *converges*, by the *ratio test*:

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{(k+1)^2}{2^{k+1}} \div \frac{k^2}{2^k} = \frac{2^k}{2^{k+1}} \cdot \frac{(k+1)^3}{k^3} \\ &= \frac{1}{2} \cdot \frac{(k+1)^3}{k^3} \to \frac{1}{2} \text{ as } k \to \infty \end{aligned}$$

$\sum \frac{1}{k!}$ *converges*, by the *ratio test*:

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{1}{(k+1)!} \div \frac{1}{k!} = \frac{k!}{(k+1)!} \\ &= \frac{1}{k+1} \to 0 \text{ as } k \to \infty \end{aligned}$$

**Examples**

$\sum \frac{k}{10^k}$ *converges*, by the *ratio test*:

$$\frac{a_{k+1}}{a_k} = \frac{k+1}{10^{k+1}} \div \frac{k}{10^k} = \frac{10^k}{10^{k+1}} \cdot \frac{k+1}{k}$$
$$= \frac{1}{10} \cdot \frac{k+1}{k} \to \frac{1}{10} \text{ as } k \to \infty$$

$\sum \frac{k^k}{k!}$ *diverges*, by the *ratio test*:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^{k+1}}{(k+1)!} \div \frac{k^k}{k!} = \frac{k!}{(k+1)!} \cdot \frac{(k+1)^{k+1}}{k^k} = \frac{(k+1)^k}{k^k}$$
$$= \left(\frac{k+1}{k}\right)^k = \left(1 + \frac{1}{k}\right)^k \to e > 1 \text{ as } k \to \infty$$

**Examples**

$\sum \frac{2^k}{3^k - 2^k}$ *converges*, by the *ratio test*:

$$\frac{a_{k+1}}{a_k} = \frac{2^{k+1}}{3^{k+1} - 2^{k+1}} \div \frac{2^k}{3^k - 2^k} = \frac{2^{k+1}}{2^k} \cdot \frac{3^k - 2^k}{3^{k+1} - 2^{k+1}}$$
$$= 2 \cdot \frac{1 - (2/3)^k}{3 - 2(2/3)^k} \to 2 \cdot \frac{1}{3} < 1 \text{ as } k \to \infty$$

$\sum \frac{1}{\sqrt{k!}}$ *converges*, by the *ratio test*:

$$\frac{a_{k+1}}{a_k} = \frac{1}{\sqrt{(k+1)!}} \div \frac{1}{\sqrt{k!}} = \sqrt{\frac{k!}{(k+1)!}}$$
$$= \sqrt{\frac{1}{k+1}} \to 0 \text{ as } k \to \infty$$

**Examples**

$\sum \frac{k^2 2^k}{k!}$ *converges*, by the *ratio test*:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^2 2^{k+1}}{(k+1)!} \div \frac{k^2 2^k}{k!} = \frac{(k+1)^2}{k^2} \cdot \frac{2^{k+1}}{2^k} \cdot \frac{k!}{(k+1)!}$$
$$= \frac{(k+1)^2}{k^2} \cdot \frac{2}{k+1} \to 0 < 1 \text{ as } k \to \infty$$

$\sum \frac{1}{2k+1}$ *inconclusive*, by the *ratio test*:

$$\frac{a_{k+1}}{a_k} = \frac{1}{2(k+1)+1} \div \frac{1}{2k+1} = \frac{2k+1}{2(k+1)+1} = \frac{2k+1}{2k+3}$$
$$= \frac{2+1/k}{2+3/k} \to 1 \text{ as } k \to \infty$$

[2ex] the series *diverges* by the *limit comparison test*:

$$\frac{1}{2k+1} \div \frac{1}{2k} = \frac{2k}{2k+1} \to 1 \text{ as } k \to \infty \text{ and } \sum \frac{1}{2k} \text{ diverges.}$$

**Outline**

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