# Lecture 24 <br> Section 11.4 Absolute and Conditional Convergence; Alternating Series 

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## Basic Series that Converge or Diverge

## Basic Series that Converge

## Geometric series:

## Basic Series that Diverge

## Basic Series that Converge or Diverge

## Basic Series that Converge

Geometric series:

p-series

Basic Series that Diverge

## Basic Series that Converge or Diverge

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Geometric series: $\quad \sum x^{k}, \quad$ if $|x|<1$
p-series:

Basic Series that Diverge

## Basic Series that Converge or Diverge

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Geometric series: $\quad \sum x^{k}, \quad$ if $|x|<1$
p-series:

if $p>1$

Basic Series that Diverge


## Basic Series that Converge or Diverge

## Basic Series that Converge

Geometric series: $\quad \sum x^{k}, \quad$ if $|x|<1$

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p \text {-series: } \quad \sum \frac{1}{k^{p}}, \quad \text { if } p>1
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Basic Series that Diverge


## Basic Series that Converge or Diverge

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Geometric series: $\quad \sum x^{k}, \quad$ if $|x|<1$

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p \text {-series: } \quad \sum \frac{1}{k^{p}}, \quad \text { if } p>1
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Basic Series that Diverge
Any series $\sum a_{k}$ for which $\lim _{k \rightarrow \infty} a_{k} \neq 0$
p-series

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Geometric series: $\quad \sum x^{k}, \quad$ if $|x|<1$

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Geometric series: $\quad \sum x^{k}, \quad$ if $|x|<1$

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p \text {-series: } \quad \sum \frac{1}{k^{p}}, \quad \text { if } p>1
$$

Basic Series that Diverge
Any series $\sum a_{k}$ for which $\lim _{k \rightarrow \infty} a_{k} \neq 0$

$$
p \text {-series: } \quad \sum \frac{1}{k^{p}}, \quad \text { if } p \leq 1
$$

## Convergence Tests (1)

## Basic Test for Convergence

Keep in Mind that,

## Examples



## Convergence Tests (1)

## Basic Test for Convergence

Keep in Mind that,

$$
\text { if } a_{k} \nrightarrow 0 \text {, then the series } \sum a_{k} \text { diverges; }
$$

therefore there is no reason to apply any special convergence test

## Examples



## Convergence Tests (1)

## Basic Test for Convergence

Keep in Mind that,
if $a_{k} \nrightarrow 0$, then the series $\sum a_{k}$ diverges;
therefore there is no reason to apply any special convergence test.

## Examples

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## Convergence Tests (1)

## Basic Test for Convergence

Keep in Mind that, if $a_{k} \nrightarrow 0$, then the series $\sum a_{k}$ diverges; therefore there is no reason to apply any special convergence test.

## Examples



## Convergence Tests (1)

## Basic Test for Convergence

Keep in Mind that, if $a_{k} \nrightarrow 0$, then the series $\sum a_{k}$ diverges; therefore there is no reason to apply any special convergence test.

## Examples

$\sum x^{k}$ with $|x| \geq 1$ (e.g, $\left.\sum(-1)^{k}\right)$ diverge since $x^{k} \nrightarrow 0$.


## Convergence Tests (1)

## Basic Test for Convergence

Keep in Mind that, if $a_{k} \nrightarrow 0$, then the series $\sum a_{k}$ diverges; therefore there is no reason to apply any special convergence test.

## Examples

$\sum x^{k}$ with $|x| \geq 1\left(\right.$ e.g, $\left.\sum(-1)^{k}\right)$ diverge since $x^{k} \nrightarrow 0$.
$\sum \frac{k}{k+1}$ diverges since $\frac{k}{k+1} \rightarrow 1 \neq 0$.


## Convergence Tests (1)

## Basic Test for Convergence

Keep in Mind that,

$$
\text { if } a_{k} \nrightarrow 0 \text {, then the series } \sum a_{k} \text { diverges; }
$$

therefore there is no reason to apply any special convergence test.

## Examples

$$
\sum x^{k} \text { with }|x| \geq 1\left(\text { e.g, } \sum(-1)^{k}\right) \text { diverge since } x^{k} \nrightarrow 0 \text {. }
$$

$$
\sum \frac{k}{k+1} \text { diverges since } \frac{k}{k+1} \rightarrow 1 \neq 0
$$

$\sum\left(1-\frac{1}{k}\right)^{k}$ diverges since $a_{k}=\left(1-\frac{1}{k}\right)^{k} \rightarrow e^{-1} \neq 0$.

## Convergence Tests (2)

## Comparison Tests

Rational terms are most easily handled by basic comparison or limit comparison with $p$-series

## Basic Comparison Test



## Convergence Tests (2)

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Rational terms are most easily handled by basic comparison or limit comparison with $p$-series $\sum 1 / k^{p}$

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## Comparison Tests

Rational terms are most easily handled by basic comparison or limit comparison with $p$-series $\sum 1 / k^{p}$

## Basic Comparison Test

$\sum \frac{1}{2 k^{3}+1}$ converges by comparison with $\sum \frac{1}{k^{3}}$

converges by comparison with


## Convergence Tests (2)

## Comparison Tests

Rational terms are most easily handled by basic comparison or limit comparison with $p$-series $\sum 1 / k^{p}$

## Basic Comparison Test

$$
\begin{aligned}
& \sum \frac{1}{2 k^{3}+1} \text { converges by comparison with } \sum \frac{1}{k^{3}} \\
& \sum \frac{k^{3}}{k^{5}+4 k^{4}+7} \text { converges by comparison with } \sum \frac{1}{k^{2}}
\end{aligned}
$$



## Convergence Tests (2)

## Comparison Tests

Rational terms are most easily handled by basic comparison or limit comparison with $p$-series $\sum 1 / k^{p}$

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$\sum \frac{1}{2 k^{3}+1}$ converges by comparison with $\sum \frac{1}{k^{3}}$
$\sum \frac{k^{3}}{k^{5}+4 k^{4}+7}$ converges by comparison with $\sum \frac{1}{k^{2}}$
$\sum \frac{1}{k^{3}-k^{2}}$ converges by comparison with $\sum \frac{2}{k^{3}}$
diverges by comparison with


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$\sum \frac{1}{k^{3}-k^{2}}$ converges by comparison with $\sum \frac{2}{k^{3}}$
$\sum \frac{1}{3 k+1}$ diverges by comparison with $\sum \frac{1}{3(k+1)}$


## Convergence Tests (2)

## Comparison Tests

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$\sum \frac{1}{k^{3}-k^{2}}$ converges by comparison with $\sum \frac{2}{k^{3}}$
$\sum \frac{1}{3 k+1}$ diverges by comparison with $\sum \frac{1}{3(k+1)}$
$\sum \frac{1}{\ln (k+6)}$ diverges by comparison with $\sum \frac{1}{k+6}$

## Convergence Tests (2)

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Rational terms are most easily handled by basic comparison or limit comparison with $p$-series $\sum 1 / k^{p}$

## Limit Comparison Test


$\square$

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Rational terms are most easily handled by basic comparison or limit comparison with $p$-series $\sum 1 / k^{p}$

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$\sum \frac{1}{k^{3}-1}$ converges by comparison with $\sum \frac{1}{k^{3}}$.


## Convergence Tests (2)

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Rational terms are most easily handled by basic comparison or limit comparison with $p$-series $\sum 1 / k^{p}$

## Limit Comparison Test

$\sum \frac{1}{k^{3}-1}$ converges by comparison with $\sum \frac{1}{k^{3}}$.
$\sum \frac{3 k^{2}+2 k+1}{k^{3}+1}$ diverges by comparison with $\sum \frac{3}{k}$


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Rational terms are most easily handled by basic comparison or limit comparison with $p$-series $\sum 1 / k^{p}$

## Limit Comparison Test

$\sum \frac{1}{k^{3}-1}$ converges by comparison with $\sum \frac{1}{k^{3}}$.
$\sum \frac{3 k^{2}+2 k+1}{k^{3}+1}$ diverges by comparison with $\sum \frac{3}{k}$
$\sum \frac{5 \sqrt{k}+100}{2 k^{2} \sqrt{k}-9 \sqrt{k}}$ converges by comparison with $\sum \frac{5}{2 k^{2}}$

## Convergence Tests (3)

## Root Test and Ratio Test

The root test is used only if powers are involved

Root Test

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$\sum \frac{k^{2}}{2^{k}}$

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\left(a_{k}\right)^{1 / k}=\frac{1}{2} \cdot\left[k^{1 / k}\right]^{2} \rightarrow \frac{1}{2} \cdot 1
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& \sum \frac{1}{(\ln k)^{k}} \text { converges: }\left(a_{k}\right)^{1 / k}=\frac{1}{\ln k} \rightarrow 0
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$\sum \frac{1}{(\ln k)^{k}}$ converges: $\left(a_{k}\right)^{1 / k}=\frac{1}{\ln k} \rightarrow 0$
$\sum\left(1-\frac{1}{k}\right)^{k^{2}}$ converges: $\left(a_{k}\right)^{1 / k}=\left(1+\frac{(-1)}{k}\right)^{k} \rightarrow e^{-1}$

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## Convergence Tests (4)

## Root Test and Ratio Test

The ratio test is effective with factorials and with combinations of powers and factorials.

Ratio Comparison Test


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Ratio Comparison Test
$\sum \frac{k^{2}}{2^{k}}$

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\frac{a_{k+1}}{a_{k}}=\frac{1}{2} \cdot \frac{(k+1)^{2}}{k^{2}} \rightarrow \frac{1}{2}
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& \sum \frac{1}{k!} \text { converges: } \frac{a_{k+1}}{a_{k}}=\frac{1}{k+1} \rightarrow 0
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& \sum \frac{k}{10^{k}} \text { converges: } \frac{a_{k+1}}{a_{k}}=\frac{1}{10} \cdot \frac{k+1}{k} \rightarrow \frac{1}{10}
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& \sum \frac{k^{k}}{k!} \text { diverges: } \frac{a_{k+1}}{a_{k}}=\left(1+\frac{1}{k}\right)^{k} \rightarrow e
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& \sum \frac{k^{k}}{k!} \text { diverges: } \frac{a_{k+1}}{a_{k}}=\left(1+\frac{1}{k}\right)^{k} \rightarrow e \\
& \sum \frac{2^{k}}{3^{k}-2^{k}} \text { converges: } \frac{a_{k+1}}{a_{k}}=2 \cdot \frac{1-(2 / 3)^{k}}{3-2(2 / 3)^{k}} \rightarrow 2 \cdot \frac{1}{3}
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& \sum \frac{k^{k}}{k!} \text { diverges: } \frac{a_{k+1}}{a_{k}}=\left(1+\frac{1}{k}\right)^{k} \rightarrow e \\
& \sum \frac{2^{k}}{3^{k}-2^{k}} \text { converges: } \frac{a_{k+1}}{a_{k}}=2 \cdot \frac{1-(2 / 3)^{k}}{3-2(2 /)^{k}} \rightarrow 2 \cdot \frac{1}{3} \\
& \sum \frac{1}{\sqrt{k!}} \text { converges: } \frac{a_{k+1}}{a_{k}}=\sqrt{\frac{1}{k+1}} \rightarrow 0
\end{aligned}
$$

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& \sum \frac{1}{k!} \text { converges: } \frac{a_{k+1}}{a_{k}}=\frac{1}{k+1} \rightarrow 0 \\
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& \sum \frac{1}{\sqrt{k!}} \text { converges: } \frac{a_{k+1}}{a_{k}}=\sqrt{\frac{1}{k+1}} \rightarrow 0
\end{aligned}
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## Absolute Convergence

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$\square$


Alternating $p$-Series with $p>1$

Geometric Series with $-1<x<1$

## Absolute Convergence

Absolute Convergence
A series $\sum a_{k}$ is said to converge absolutely if $\sum\left|a_{k}\right|$ converges.

Alternating $p$-Series with $p>1$

Geometric Series with $-1<x<1$

## Absolute Convergence

Absolute Convergence
A series $\sum a_{k}$ is said to converge absolutely if $\sum\left|a_{k}\right|$ converges.

$$
\text { if } \sum\left|a_{k}\right| \text { converges, then } \sum a_{k} \text { converges. }
$$

Alternating $p$-Series with $p>1$

$\sum$

Geometric Series with $-1<x<1$


## Absolute Convergence

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A series $\sum a_{k}$ is said to converge absolutely if $\sum\left|a_{k}\right|$ converges.
if $\sum\left|a_{k}\right|$ converges, then $\sum a_{k}$ converges.
i.e., absolutely convergent series are convergent.

Alternating $p$-Series with $p>1$


Geometric Series with $-1<x<1$

converges

## Absolute Convergence

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Alternating $p$-Series with $p>1$


Geometric Series with $-1<x<1$
$\sum(-1)^{j(k)} x^{k},-1<x<1$, converge absolutely because $\sum|x|^{k}$ converges.

## Absolute Convergence

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A series $\sum a_{k}$ is said to converge absolutely if $\sum\left|a_{k}\right|$ converges.
if $\sum\left|a_{k}\right|$ converges, then $\sum a_{k}$ converges.
i.e., absolutely convergent series are convergent.

Alternating $p$-Series with $p>1$

$$
\begin{aligned}
& \sum \frac{(-1)^{k}}{k^{p}}, p>1, \text { converge absolutely because } \sum \frac{1}{k^{p}} \text { converges. } \\
& \Rightarrow \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2}}=1-\frac{1}{2^{2}}+\frac{1}{3^{2}}-\frac{1}{4^{2}}-\cdots \text { converge absolutely. }
\end{aligned}
$$

Geometric Series with $-1<x<1$
$\sum(-1)^{j(k)} x^{k},-1<x<1$, converge absolutely because $\sum|x|^{k}$ converges.
$\Rightarrow \quad 1-\frac{1}{2}-\frac{1}{2^{2}}+\frac{1}{2^{3}}-\frac{1}{2^{4}}+\frac{1}{2^{5}}+\frac{1}{2^{6}}-\cdots$ converge absolutely.

## Conditional Convergence

## Conditional Convergence

A series $\sum a_{k}$ is said to converge conditionally if $\sum a_{k}$ converges while $\sum\left|a_{k}\right|$ diverges.

## Alternating $p$-Series with $0<p \leq 1$


$\qquad$

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Alternating $p$-Series with $0<p \leq 1$
$\sum \frac{(-1)^{k}}{k^{p}}, 0<p \leq 1$, converge conditionally because $\sum \frac{1}{k^{p}}$ diverges.


## Conditional Convergence

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A series $\sum a_{k}$ is said to converge conditionally if $\sum a_{k}$ converges while $\sum\left|a_{k}\right|$ diverges.

Alternating $p$-Series with $0<p \leq 1$
$\sum \frac{(-1)^{k}}{k^{p}}, 0<p \leq 1$, converge conditionally because $\sum \frac{1}{k^{p}}$ diverges.
$\Rightarrow \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\cdots$ converge conditionally.

## Alternating Series

## Alternating Series

## Let $\left\{a_{k}\right\}$ be a sequence of positive numbers.



## Alternating Series Test

## Alternating $p$-Series with $p>0$

## Alternating Series

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Let $\left\{a_{k}\right\}$ be a sequence of positive numbers.

is called an alternating series.
Alternating Series Test
Let $\left\{a_{k}\right\}$ be a decreasing sequence of positive numbers

Alternating $p$-Series with $p>0$

## Alternating Series

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$$
\sum(-1)^{k} a_{k}=a_{0}-a_{1}+a_{2}-a_{3}+a_{4}-\cdots
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Alternating $p$-Series with $p>0$
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\text { If } a_{k} \rightarrow 0, \quad \text { then } \sum(-1)^{k} a_{k} \text { converges. }
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Alternating $p$-Series with $p>0$


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$\sum \frac{(-1)^{k}}{k^{p}}, p>0$, converge since 0 for $\forall x>0$, and


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## Alternating $p$-Series with $p>0$

$$
\begin{aligned}
& \sum \frac{(-1)^{k}}{k^{p}}, p>0, \text { converge since } f(x)=\frac{1}{x^{p}} \text { is decreasing, i.e., } \\
& f^{\prime}(x)=-\frac{p}{x^{p+1}}>0 \text { for } \forall x>0 \text {, and } \lim f(x)=0 \\
& \Rightarrow \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\cdots \text { converge conditionally. }
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## Examples



## Examples



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$\sum \frac{(-1)^{k}}{2 k+1}$, converge since $f(x)=\frac{1}{2 x+1}$ is decreasing, i.e., $f^{\prime}(x)=-\frac{2}{(2 x+1)^{2}}>0$ for $\forall x>0$,
$\sum \frac{(-1)^{k} k}{k^{2}+10}$, converge since $f(x)=\frac{x}{x^{2}+10}$ is decreasing, i.e.,
$f^{\prime}(x)=-\frac{x^{2}-10}{\left(x^{2}+10\right)^{2}}>0$, for $\forall x>\sqrt{10}$,

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## An Estimate for Alternating Series



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Let $\left\{a_{k}\right\}$ be a decreasing sequence of positive numbers that tends

consecutive partial sums $S_{n}, S_{n+1}$,


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Let $\left\{a_{k}\right\}$ be a decreasing sequence of positive numbers that tends to 0 and let $L=\sum_{k=0}^{\infty}(-1)^{k} a_{k}$. Then the sum $L$ lies between
consecutive partial sums $s_{n}, s_{n+1}$,

$$
s_{n}<L<s_{n+1} \text {, if } n \text { is odd; } s_{n+1}<L<s_{n} \text {, if } n \text { is even. }
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and thus $s_{n}$ approximates $L$ to within $a_{n+1}$

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$$
\left|L-s_{n}\right|<a_{n+1} .
$$

## Example

Find $s_{n}$ to approximate $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=1-\frac{1}{2}+\frac{1}{3} \cdots$ within $10^{-2}$
$k=1$


## Example

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Set $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{k+1}$. For $\left|L-s_{n}\right|<10^{-2}$, we want $a_{n+1}=\frac{1}{(n+1)+1}<10^{-2} \Rightarrow n+2>10^{2} \Rightarrow n>98$.

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From the estimate
$\left|L-S_{99}\right|<a_{100}=\frac{1}{101} \approx 0.00991$

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Then $n=99$ and the 99th partial sum $s_{100}$ is

$$
s_{99}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{99}-\frac{1}{100} \approx 0.6882 .
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$$
\mathrm{s}_{99} \approx 0.6882<\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=\ln 2<0.6981 \approx s_{100}
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## Example

$$
\text { Find } s_{n} \text { to approximate } \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2 k+1)!}=1-\frac{1}{3!}+\frac{1}{5!}
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For $\left|L-s_{n}\right|<10^{-2}$, we want

$$
a_{n+1}=\frac{1}{(2(n+1)+1)!}<10^{-2} \quad \Rightarrow \quad n \geq 1
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Then $n=1$ and the 2 nd partial sum $s_{2}$ is

$$
s_{1}=1-\frac{1}{3!} \approx 0.8333
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From the estimate


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$$

we conclude that

$$
s_{1} \approx 0.8333<\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2 k+1)!}=\sin 1<0.8416 \approx s_{2}
$$

## Why Absolute Convergence Matters: Rearrangements (1)

Rearrangement of Absolute Convergence Series


Theorem
$\qquad$

## Why Absolute Convergence Matters: Rearrangements (1)

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$$
\sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{k}}=1-\frac{1}{2}+\frac{1}{2^{2}}-\frac{1}{2^{3}}+\frac{1}{2^{4}}-\frac{1}{2^{5}}
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& \text { Rearrangement } 1+\frac{1}{2^{2}}-\frac{1}{2}+\frac{1}{2^{4}}+\frac{1}{2^{6}}-\frac{1}{2^{3}}+\frac{1}{2^{8}}+\frac{1}{2^{10}}-\frac{1}{2^{5}} \cdots
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\end{aligned}
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## Theorem

All rearrangements of an absolutely convergent series converge absolutely to the same sum.

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Rearrangement of Absolute Convergence Series

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$$

$$
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Rearrangement $1+\frac{1}{2^{2}}-\frac{1}{2}+\frac{1}{2^{4}}+\frac{1}{2^{6}}-\frac{1}{2^{3}}+\frac{1}{2^{8}}+\frac{1}{2^{10}}-\frac{1}{2^{5}} \cdots=\frac{2}{3}$

## Theorem

All rearrangements of an absolutely convergent series converge absolutely to the same sum.

## Why Absolute Convergence Matters: Rearrangements (2)

Rearrangement of Conditional Convergence Series


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$$
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}
$$

## Why Absolute Convergence Matters: Rearrangements (2)

Rearrangement of Conditional Convergence Series

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\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots
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## Why Absolute Convergence Matters: Rearrangements (2)

Rearrangement of Conditional Convergence Series

$$
\begin{aligned}
& \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\ln 2 \text { conditionally } \\
& \text { Rearrangement } 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6} \cdots ?=\ln 2
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Rearrangement of Conditional Convergence Series

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\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\ln 2 \text { conditionally }
$$

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$$

## Multiply the original series by $\frac{1}{2}$



## Why Absolute Convergence Matters: Rearrangements (2)

## Rearrangement of Conditional Convergence Series

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\begin{aligned}
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\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}
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\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}+\cdots
$$

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\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}+\cdots=\frac{1}{2} \ln 2
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Adding the two series, we get the rearrangement


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$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\ln 2$ conditionally
Rearrangement $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6} \cdots \neq \ln 2$
Multiply the original series by $\frac{1}{2}$

$$
\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}+\cdots=\frac{1}{2} \ln 2
$$

Adding the two series, we get the rearrangement

$$
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}+\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}
$$

## Why Absolute Convergence Matters: Rearrangements (2)

Rearrangement of Conditional Convergence Series
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Multiply the original series by $\frac{1}{2}$

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\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}+\cdots=\frac{1}{2} \ln 2
$$

Adding the two series, we get the rearrangement

$$
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}+\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots
$$

## Why Absolute Convergence Matters: Rearrangements (2)

## Rearrangement of Conditional Convergence Series

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\ln 2$ conditionally
Rearrangement $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6} \cdots \neq \ln 2$
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Remark

- A series that is only conditionally convergent can be rearranged to converge to any number we please. It can also be arranged to diverge to $+\infty$ or $-\infty$, or even to oscillate between any two bounds we choose.


## Why Absolute Convergence Matters: Rearrangements (2)

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## Outline

- Convergence Tests
- Absolute Convergence
- Absolute Convergence
- Alternating Series
- Rearrangements

