Lecture 24 Section 11.4 Absolute and Conditional Convergence; Alternating Series

Jiwen He

1 Convergence Tests

Basic Series that Converge or Diverge

Basic Series that Converge

$$\text{Geometric series:} \quad \sum x^k, \quad \text{if } |x| < 1$$
$$p\text{-series:} \quad \sum \frac{1}{k^p}, \quad \text{if } p > 1$$

Basic Series that Diverge

$$\text{Any series } \sum a_k \quad \text{for which } \lim_{k\to\infty} a_k \neq 0$$
$$p\text{-series:} \quad \sum \frac{1}{k^p}, \quad \text{if } p \leq 1$$

Convergence Tests (1)

Basic Test for Convergence
Keep in Mind that, if $a_k \nrightarrow 0$, then the series $\sum a_k$ *diverges*; therefore there is no reason to apply any special convergence test.

Examples 1. $\sum x^k$ with $|x| \geq 1$ (e.g, $\sum(-1)^k$) *diverge* since $x^k \nrightarrow 0$. [1ex] $\displaystyle\sum \frac{k}{k+1}$ *diverges* since $\frac{k}{k+1} \to 1 \neq 0$. [1ex] $\displaystyle\sum \left(1 - \frac{1}{k}\right)^k$ *diverges* since $a_k = \left(1 - \frac{1}{k}\right)^k \to e^{-1} \neq 0$.

Convergence Tests (2)

Comparison Tests
Rational terms are most easily handled by *basic comparison* or *limit comparison* with p-series $\sum 1/k^p$

Basic Comparison Test

$\sum \frac{1}{2k^3+1}$ converges by comparison with $\sum \frac{1}{k^3}$ $\sum \frac{k^3}{k^5+4k^4+7}$ converges by comparison with $\sum \frac{1}{k^2}$ $\sum \frac{1}{k^3-k^2}$ converges by comparison with $\sum \frac{2}{k^3}$ $\sum \frac{1}{3k+1}$ diverges by comparison with $\sum \frac{1}{3(k+1)}$ $\sum \frac{1}{\ln(k+6)}$ diverges by comparison with $\sum \frac{1}{k+6}$

Limit Comparison Test
$\sum \frac{1}{k^3-1}$ converges by comparison with $\sum \frac{1}{k^3}$. $\sum \frac{3k^2+2k+1}{k^3+1}$ diverges by comparison with $\sum \frac{3}{k}$ $\sum \frac{5\sqrt{k}+100}{2k^2\sqrt{k}-9\sqrt{k}}$ converges by comparison with $\sum \frac{5}{2k^2}$

Convergence Tests (3)

Root Test and Ratio Test
The *root test* is used only if *powers* are involved.

Root Test
$\sum \frac{k^2}{2^k}$ *converges*: $(a_k)^{1/k} = \frac{1}{2} \cdot \left[k^{1/k}\right]^2 \to \frac{1}{2} \cdot 1$ $\sum \frac{1}{(\ln k)^k}$ *converges*: $(a_k)^{1/k} = \frac{1}{\ln k} \to 0$ $\sum \left(1 - \frac{1}{k}\right)^{k^2}$ *converges*: $(a_k)^{1/k} = \left(1 + \frac{(-1)}{k}\right)^k \to e^{-1}$

Convergence Tests (4)
Root Test and Ratio Test
The *ratio test* is effective with *factorials* and with combinations of powers and factorials.

Ratio Comparison Test
$\sum \frac{k^2}{2^k}$ *converges*: $\frac{a_{k+1}}{a_k} = \frac{1}{2} \cdot \frac{(k+1)^2}{k^2} \to \frac{1}{2}$ $\sum \frac{1}{k!}$ *converges*: $\frac{a_{k+1}}{a_k} = \frac{1}{k+1} \to 0$ $\sum \frac{k}{10^k}$ *converges*: $\frac{a_{k+1}}{a_k} = \frac{1}{10} \cdot \frac{k+1}{k} \to \frac{1}{10}$ $\sum \frac{k^k}{k!}$ *diverges*: $\frac{a_{k+1}}{a_k} = \left(1 + \frac{1}{k}\right)^k \to e$ $\sum \frac{2^k}{3^k - 2^k}$ *converges*: $\frac{a_{k+1}}{a_k} = 2 \cdot \frac{1-(2/3)^k}{3-2(2/3)^k} \to 2 \cdot \frac{1}{3}$ $\sum \frac{1}{\sqrt{k!}}$ *converges*: $\frac{a_{k+1}}{a_k} = \sqrt{\frac{1}{k+1}} \to 0$

2 Absolute Convergence

2.1 Absolute Convergence

Absolute Convergence

Absolute Convergence
A series $\sum a_k$ is said to *converge absolutely* if $\sum |a_k|$ converges.

if $\sum |a_k|$ converges, then $\sum a_k$ converges.
i.e., absolutely convergent series are convergent.

Alternating p-Series with $p > 1$
$\displaystyle\sum \frac{(-1)^k}{k^p}$, $p > 1$, *converge absolutely* because $\displaystyle\sum \frac{1}{k^p}$ converges. $\Rightarrow$ $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \cdots$ *converge absolutely.*

Geometric Series with $-1 < x < 1$
$\sum (-1)^{j(k)} x^k$, $-1 < x < 1$, *converge absolutely* because $\sum |x|^k$ converges.
$\Rightarrow \quad \displaystyle 1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} - \cdots$ *converge absolutely.*

Conditional Convergence

Conditional Convergence
A series $\sum a_k$ is said to *converge conditionally* if $\sum a_k$ converges while $\sum |a_k|$ diverges.

Alternating p-Series with $0 < p \leq 1$
$\displaystyle\sum \frac{(-1)^k}{k^p}$, $0 < p \leq 1$, *converge conditionally* because $\displaystyle\sum \frac{1}{k^p}$ diverges. $\Rightarrow$ $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \cdots$ *converge conditionally.*

3 Alternating Series

Alternating Series
Alternating Series
Let $\{a_k\}$ be a sequence of *positive* numbers.

$$\sum (-1)^k a_k = a_0 - a_1 + a_2 - a_3 + a_4 - \cdots$$

is called an *alternating series.*

Alternating Series Test
Let $\{a_k\}$ be a *decreasing* sequence of *positive* numbers.

If $a_k \to 0$, then $\sum (-1)^k a_k$ converges.

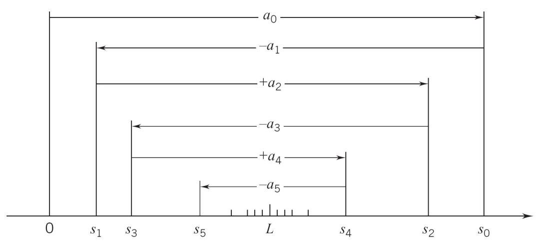
Alternating p-Series with $p > 0$
$\displaystyle\sum \frac{(-1)^k}{k^p}$, $p > 0$, *converge* since $f(x) = \dfrac{1}{x^p}$ is *decreasing*, i.e., $f'(x) = -\dfrac{p}{x^{p+1}} > 0$ for $\forall x > 0$, and $\displaystyle\lim_{x\to\infty} f(x) = 0$. $\Rightarrow$ $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \cdots$ converge *conditionally.*

Examples

$\sum \frac{(-1)^k}{2k+1}$, *converge* since $f(x) = \frac{1}{2x+1}$ is *decreasing*, i.e., $f'(x) = -\frac{2}{(2x+1)^2} > 0$ for $\forall x > 0$, and $\lim_{x\to\infty} f(x) = 0$.

$\sum \frac{(-1)^k k}{k^2+10}$, *converge* since $f(x) = \frac{x}{x^2+10}$ is *decreasing*, i.e., $f'(x) = -\frac{x^2-10}{(x^2+10)^2} > 0$, for $\forall x > \sqrt{10}$, and $\lim_{x\to\infty} f(x) = 0$.

An Estimate for Alternating Series



An Estimate for Alternating Series

Let $\{a_k\}$ be a *decreasing* sequence of *positive* numbers that tends to 0 and let $L = \sum_{k=0}^{\infty} (-1)^k a_k$. Then the sum L lies between consecutive *partial sums* s_n, s_{n+1},

$$s_n < L < s_{n+1}, \text{ if } n \text{ is odd}; \quad s_{n+1} < L < s_n, \text{ if } n \text{ is even}.$$

and thus s_n *approximates* L to within a_{n+1}

$$|L - s_n| < a_{n+1}.$$

Example

Find s_n to approximate $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} \cdots$ within 10^{-2}.

Set $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$. For $|L - s_n| < 10^{-2}$, we want

$$a_{n+1} = \frac{1}{(n+1)+1} < 10^{-2} \quad \Rightarrow \quad n+2 > 10^2 \quad \Rightarrow \quad n > 98.$$

Then $n = 99$ and the 99th partial sum s_{100} is

$$s_{99} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99} - \frac{1}{100} \approx 0.6882.$$

From the estimate

$$|L - s_{99}| < a_{100} = \frac{1}{101} \approx 0.00991.$$

we conclude that

$$s_{99} \approx 0.6882 < \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2 < 0.6981 \approx s_{100}$$

Example

Find s_n to approximate $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} \cdots$ within 10^{-2}.

For $|L - s_n| < 10^{-2}$, we want

$$a_{n+1} = \frac{1}{(2(n+1)+1)!} < 10^{-2} \quad \Rightarrow \quad n \geq 1.$$

Then $n = 1$ and the 2nd partial sum s_2 is

$$s_1 = 1 - \frac{1}{3!} \approx 0.8333$$

From the estimate

$$|L - s_1| < a_2 = \frac{1}{5!} \approx 0.0083.$$

we conclude that

$$s_1 \approx 0.8333 < \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} = \sin 1 < 0.8416 \approx s_2$$

4 Rearrangements

Why Absolute Convergence Matters: Rearrangements (1)

Rearrangement of Absolute Convergence Series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \cdots = \frac{2}{3} \textit{ absolutely}$$

$$\textit{Rearrangement } 1 + \frac{1}{2^2} - \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^6} - \frac{1}{2^3} + \frac{1}{2^8} + \frac{1}{2^{10}} - \frac{1}{2^5} \cdots ? = \; = \frac{2}{3}$$

Theorem 2. *All rearrangements of an absolutely convergent series converge absolutely to the same sum.*

Why Absolute Convergence Matters: Rearrangements (2)
Rearrangement of Conditional Convergence Series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2 \textit{ conditionally}$$

$$\textit{Rearrangement } 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} \cdots ? = \; \neq \ln 2$$

Multiply the original series by $\frac{1}{2}$

$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} + \cdots = \frac{1}{2} \ln 2$$

Adding the two series, we get the rearrangement

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2} \ln 2$$

Remark

- A series that is only *conditionally* convergent can be rearranged to converge to *any number* we please.
- It can also be arranged to *diverge* to $+\infty$ or $-\infty$, or even to oscillate between any two bounds we choose.

Outline

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