## Lecture 25

## Section 11.5 Taylor Polynomials in $x$; Taylor Series in $x$

## Jiwen He

Department of Mathematics, University of Houston
jiwenhe@math.uh.edu
http://math.uh.edu/~jiwenhe/Math1432


## Taylor Polynomials

Taylor Polynomials
The nth Taylor polynomial at 0 for a function $f$ is

Best Approximation

## Taylor Polynomials

Taylor Polynomials
The $n$th Taylor polynomial at 0 for a function $f$ is

$\square$

Best Approximation

## Taylor Polynomials

## Taylor Polynomials

The $n$th Taylor polynomial at 0 for a function $f$ is

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$P_{n}$ is the polynomial that has the same value as $f$ at 0 and the same first $n$ derivatives:
$\square$

Best Approximation

## Taylor Polynomials

## Taylor Polynomials

The $n$th Taylor polynomial at 0 for a function $f$ is

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$P_{n}$ is the polynomial that has the same value as $f$ at 0 and the same first $n$ derivatives:


Best Approximation

## Taylor Polynomials

## Taylor Polynomials

The $n$th Taylor polynomial at 0 for a function $f$ is

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$P_{n}$ is the polynomial that has the same value as $f$ at 0 and the same first $n$ derivatives:

$$
P_{n}(0)=f(0), P_{n}^{\prime}(0)=f^{\prime}(0)
$$

Best Approximation

## Taylor Polynomials

## Taylor Polynomials

The $n$th Taylor polynomial at 0 for a function $f$ is

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$P_{n}$ is the polynomial that has the same value as $f$ at 0 and the same first $n$ derivatives:

$$
\begin{equation*}
P_{n}(0)=f(0), P_{n}^{\prime}(0)=f^{\prime}(0) \tag{n}
\end{equation*}
$$

Best Approximation

## Taylor Polynomials

## Taylor Polynomials

The $n$th Taylor polynomial at 0 for a function $f$ is

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$P_{n}$ is the polynomial that has the same value as $f$ at 0 and the same first $n$ derivatives:
$P_{n}(0)=f(0), P_{n}^{\prime}(0)=f^{\prime}(0), P_{n}^{\prime \prime}(0)=f^{\prime \prime}(0)$,


Best Approximation

## Taylor Polynomials

## Taylor Polynomials

The $n$th Taylor polynomial at 0 for a function $f$ is

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$P_{n}$ is the polynomial that has the same value as $f$ at 0 and the same first $n$ derivatives:

$$
P_{n}(0)=f(0), P_{n}^{\prime}(0)=f^{\prime}(0), P_{n}^{\prime \prime}(0)=f^{\prime \prime}(0), \cdots, P_{n}^{(n)}(0)=f^{(n)}(0) .
$$

Best Approximation
$P_{n}$ provides the best local approximation of $f(x)$ near 0 by a polynomial of degree $\leq n$.

## Taylor Polynomials

## Taylor Polynomials

The $n$th Taylor polynomial at 0 for a function $f$ is

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$P_{n}$ is the polynomial that has the same value as $f$ at 0 and the same first $n$ derivatives:

$$
P_{n}(0)=f(0), P_{n}^{\prime}(0)=f^{\prime}(0), P_{n}^{\prime \prime}(0)=f^{\prime \prime}(0), \cdots, P_{n}^{(n)}(0)=f^{(n)}(0) .
$$

## Best Approximation

$P_{n}$ provides the best local approximation of $f(x)$ near 0 by a polynomial of degree $\leq n$.

## Taylor Polynomials

## Taylor Polynomials

The $n$th Taylor polynomial at 0 for a function $f$ is

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$P_{n}$ is the polynomial that has the same value as $f$ at 0 and the same first $n$ derivatives:

$$
P_{n}(0)=f(0), P_{n}^{\prime}(0)=f^{\prime}(0), P_{n}^{\prime \prime}(0)=f^{\prime \prime}(0), \cdots, P_{n}^{(n)}(0)=f^{(n)}(0)
$$

## Best Approximation

$P_{n}$ provides the best local approximation of $f(x)$ near 0 by a polynomial of degree $\leq n$.

$$
P_{0}(x)=f(0),
$$



## Taylor Polynomials

## Taylor Polynomials

The $n$th Taylor polynomial at 0 for a function $f$ is

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$P_{n}$ is the polynomial that has the same value as $f$ at 0 and the same first $n$ derivatives:

$$
P_{n}(0)=f(0), P_{n}^{\prime}(0)=f^{\prime}(0), P_{n}^{\prime \prime}(0)=f^{\prime \prime}(0), \cdots, P_{n}^{(n)}(0)=f^{(n)}(0)
$$

## Best Approximation

$P_{n}$ provides the best local approximation of $f(x)$ near 0 by a polynomial of degree $\leq n$.

$$
\begin{aligned}
& P_{0}(x)=f(0) \\
& P_{1}(x)=f(0)+f^{\prime}(0) x,
\end{aligned}
$$

$$
P_{2}(x)=f(0)
$$

## Taylor Polynomials

## Taylor Polynomials

The $n$th Taylor polynomial at 0 for a function $f$ is

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$P_{n}$ is the polynomial that has the same value as $f$ at 0 and the same first $n$ derivatives:

$$
P_{n}(0)=f(0), P_{n}^{\prime}(0)=f^{\prime}(0), P_{n}^{\prime \prime}(0)=f^{\prime \prime}(0), \cdots, P_{n}^{(n)}(0)=f^{(n)}(0)
$$

## Best Approximation

$P_{n}$ provides the best local approximation of $f(x)$ near 0 by a polynomial of degree $\leq n$.

$$
\begin{aligned}
& P_{0}(x)=f(0) \\
& P_{1}(x)=f(0)+f^{\prime}(0) x, \\
& P_{2}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2} .
\end{aligned}
$$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$



Taylor Polynomials of $f(x)=e^{x}$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
f(x)=e^{x},
$$



Taylor Polynomials of $f(x)=e^{x}$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
f(x)=e^{x}, \quad f^{\prime}(x)=e^{x},
$$



## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
f(x)=e^{x}, \quad f^{\prime}(x)=e^{x}, \quad f^{\prime \prime}(x)=e^{x},
$$



## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$$
f(x)=e^{x}, \quad f^{\prime}(x)=e^{x}, \quad f^{\prime \prime}(x)=e^{x}, \quad \cdots, \quad f^{(n)}(x)=e^{x} ;
$$



## Taylor Polynomials of $f(x)=e^{x}$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$$
\begin{gathered}
f(x)=e^{x}, \quad f^{\prime}(x)=e^{x}, \quad f^{\prime \prime}(x)=e^{x}, \quad \cdots, \quad f^{(n)}(x)=e^{x} ; \\
f(0)=1, \quad f^{\prime}(0)=1,
\end{gathered}
$$



## Taylor Polynomials of $f(x)=e^{x}$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$$
\begin{array}{rlll}
f(x)=e^{x}, & f^{\prime}(x)=e^{x}, & f^{\prime \prime}(x)=e^{x}, & \cdots,
\end{array} \quad f^{(n)}(x)=e^{x} ;
$$



## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
\begin{array}{rlll}
f(x)=e^{x}, & f^{\prime}(x)=e^{x}, & f^{\prime \prime}(x)=e^{x}, & \cdots,
\end{array} f^{(n)}(x)=e^{x} ;
$$



## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
\begin{aligned}
& f(x)=e^{x}, \quad f^{\prime}(x)=e^{x}, \quad f^{\prime \prime}(x)=e^{x}, \quad \cdots, \quad f^{(n)}(x)=e^{x} ; \\
& f(0)=1, \quad f^{\prime}(0)=1, \quad f^{\prime \prime}(0)=1, \quad \cdots, \quad f^{(n)}(0)=1 .
\end{aligned}
$$



## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$$
\begin{aligned}
& f(x)=e^{x}, \quad f^{\prime}(x)=e^{x}, \quad f^{\prime \prime}(x)=e^{x}, \quad \cdots, \quad f^{(n)}(x)=e^{x} ; \\
& f(0)=1, \quad f^{\prime}(0)=1, \quad f^{\prime \prime}(0)=1, \quad \cdots, \quad f^{(n)}(0)=1 .
\end{aligned}
$$



## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$$
\begin{aligned}
& f(x)=e^{x}, \quad f^{\prime}(x)=e^{x}, \quad f^{\prime \prime}(x)=e^{x}, \quad \cdots, \quad f^{(n)}(x)=e^{x} ; \\
& f(0)=1, \quad f^{\prime}(0)=1, \quad f^{\prime \prime}(0)=1, \quad \cdots, \quad f^{(n)}(0)=1 .
\end{aligned}
$$



Taylor Polynomials of $f(x)=e^{x}$

$$
P_{0}(x)=1,
$$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$$
\begin{array}{clll}
f(x)=e^{x}, & f^{\prime}(x)=e^{x}, & f^{\prime \prime}(x)=e^{x}, & \cdots,
\end{array} \quad f^{(n)}(x)=e^{x} ; ~ 子 ~ f(0)=1, \quad f^{\prime}(0)=1, \quad f^{\prime \prime}(0)=1, \quad \cdots, \quad f^{(n)}(0)=1 .
$$



Taylor Polynomials of $f(x)=e^{x}$

$$
\begin{aligned}
& P_{0}(x)=1 \\
& P_{1}(x)=1+x,
\end{aligned}
$$



## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$$
\begin{array}{clll}
f(x)=e^{x}, & f^{\prime}(x)=e^{x}, & f^{\prime \prime}(x)=e^{x}, & \cdots,
\end{array} \quad f^{(n)}(x)=e^{x} ; ~ 子 ~ f(0)=1, \quad f^{\prime}(0)=1, \quad f^{\prime \prime}(0)=1, \quad \cdots, \quad f^{(n)}(0)=1 .
$$



Taylor Polynomials of $f(x)=e^{x}$

$$
\begin{aligned}
& P_{0}(x)=1, \\
& P_{1}(x)=1+x, \\
& P_{2}(x)=1+x+\frac{x^{2}}{2!},
\end{aligned}
$$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$$
\begin{aligned}
& f(x)=e^{x}, \quad f^{\prime}(x)=e^{x}, \quad f^{\prime \prime}(x)=e^{x}, \quad \cdots, \quad f^{(n)}(x)=e^{x} ; \\
& f(0)=1, \quad f^{\prime}(0)=1, \quad f^{\prime \prime}(0)=1, \quad \cdots, \quad f^{(n)}(0)=1 .
\end{aligned}
$$



Taylor Polynomials of $f(x)=e^{x}$

$$
\begin{aligned}
& P_{0}(x)=1, \\
& P_{1}(x)=1+x, \\
& P_{2}(x)=1+x+\frac{x^{2}}{2!}, \\
& P_{3}(x)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!},
\end{aligned}
$$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$$
\begin{aligned}
& f(x)=e^{x}, \quad f^{\prime}(x)=e^{x}, \quad f^{\prime \prime}(x)=e^{x}, \quad \cdots, \quad f^{(n)}(x)=e^{x} ; \\
& f(0)=1, \quad f^{\prime}(0)=1, \quad f^{\prime \prime}(0)=1, \quad \cdots, \quad f^{(n)}(0)=1 .
\end{aligned}
$$



Taylor Polynomials of $f(x)=e^{x}$

$$
\begin{aligned}
& P_{0}(x)=1, \\
& P_{1}(x)=1+x, \\
& P_{2}(x)=1+x+\frac{x^{2}}{2!}, \\
& P_{3}(x)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!},
\end{aligned}
$$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

$$
\begin{aligned}
& f(x)=e^{x}, \quad f^{\prime}(x)=e^{x}, \quad f^{\prime \prime}(x)=e^{x}, \quad \cdots, \quad f^{(n)}(x)=e^{x} ; \\
& f(0)=1, \quad f^{\prime}(0)=1, \quad f^{\prime \prime}(0)=1, \quad \cdots, \quad f^{(n)}(0)=1 .
\end{aligned}
$$



Taylor Polynomials of $f(x)=e^{x}$

$$
\begin{aligned}
& P_{0}(x)=1, \\
& P_{1}(x)=1+x, \\
& P_{2}(x)=1+x+\frac{x^{2}}{2!}, \\
& P_{3}(x)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}, \\
& \vdots \\
& P_{n}(x)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots+\frac{x^{n}}{n!} .
\end{aligned}
$$

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$f(x)=\sin x, f^{\prime}(x)=\cos x$


## Taylor Polynomials of $f(x)=\sin x$



## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$f(x)=\sin x$,


## Taylor Polynomials of $f(x)=\sin x$



## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$f(x)=\sin x, f^{\prime}(x)=\cos x$,

## Taylor Polynomials of $f(x)=\sin x$



## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x$,


## Taylor Polynomials of $f(x)=\sin x$



## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x, f^{\prime \prime \prime}(x)=-\cos x, \cdots$ $f(0)=0$,

## Taylor Polynomials of $f(x)=\sin x$



## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x, f^{\prime \prime \prime}(x)=-\cos x, \cdots$ $f(0)=0$,

## Taylor Polynomials of $f(x)=\sin x$



## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x, f^{\prime \prime \prime}(x)=-\cos x, \cdots$ $f(0)=0, f^{\prime}(0)=1$,

## Taylor Polynomials of $f(x)=\sin x$



## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
\begin{aligned}
& f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x, f^{\prime \prime \prime}(x)=-\cos x, \cdots ; \\
& f(0)=0, f^{\prime}(0)=1, f^{\prime \prime}(0)=0, f^{\prime \prime \prime}(0)=-1, f^{(4)}(0)=0
\end{aligned}
$$

## Taylor Polynomials of $f(x)=\sin x$



## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
\begin{aligned}
& f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x, f^{\prime \prime \prime}(x)=-\cos x, \cdots ; \\
& f(0)=0, f^{\prime}(0)=1, f^{\prime \prime}(0)=0, f^{\prime \prime \prime}(0)=-1, f^{(4)}(0)=0, \cdots ;
\end{aligned}
$$

Taylor Polynomials of $f(x)=\sin x$


## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
\begin{aligned}
& f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x, f^{\prime \prime \prime}(x)=-\cos x, \cdots ; \\
& f(0)=0, f^{\prime}(0)=1, f^{\prime \prime}(0)=0, f^{\prime \prime \prime}(0)=-1, f^{(4)}(0)=0, \cdots ;
\end{aligned}
$$

## Taylor Polynomials of $f(x)=\sin x$



## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
\begin{aligned}
& f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x, f^{\prime \prime \prime}(x)=-\cos x, \cdots ; \\
& f(0)=0, f^{\prime}(0)=1, f^{\prime \prime}(0)=0, f^{\prime \prime \prime}(0)=-1, f^{(4)}(0)=0, \cdots ;
\end{aligned}
$$

## Taylor Polynomials of $f(x)=\sin x$

$$
P_{0}(x)=0,
$$



## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
\begin{aligned}
& f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x, f^{\prime \prime \prime}(x)=-\cos x, \cdots ; \\
& f(0)=0, f^{\prime}(0)=1, f^{\prime \prime}(0)=0, f^{\prime \prime \prime}(0)=-1, f^{(4)}(0)=0, \cdots ;
\end{aligned}
$$

## Taylor Polynomials of $f(x)=\sin x$

$$
\begin{aligned}
& P_{0}(x)=0, \\
& P_{1}(x)=P_{2}(x)=x,
\end{aligned}
$$



## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
\begin{aligned}
& f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x, f^{\prime \prime \prime}(x)=-\cos x, \cdots ; \\
& f(0)=0, f^{\prime}(0)=1, f^{\prime \prime}(0)=0, f^{\prime \prime \prime}(0)=-1, f^{(4)}(0)=0, \cdots ;
\end{aligned}
$$

Taylor Polynomials of $f(x)=\sin x$

$$
\begin{aligned}
& P_{0}(x)=0, \\
& P_{1}(x)=P_{2}(x)=x, \\
& P_{3}(x)=P_{4}(x)=x-\frac{x^{3}}{3!},
\end{aligned}
$$

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
\begin{aligned}
& f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x, f^{\prime \prime \prime}(x)=-\cos x, \cdots ; \\
& f(0)=0, f^{\prime}(0)=1, f^{\prime \prime}(0)=0, f^{\prime \prime \prime}(0)=-1, f^{(4)}(0)=0, \cdots ;
\end{aligned}
$$

Taylor Polynomials of $f(x)=\sin x$

$$
\begin{aligned}
& P_{0}(x)=0, \\
& P_{1}(x)=P_{2}(x)=x, \\
& P_{3}(x)=P_{4}(x)=x-\frac{x^{3}}{3!}, \\
& P_{5}(x)=P_{6}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!},
\end{aligned}
$$

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$

$$
\begin{aligned}
& f(x)=\sin x, f^{\prime}(x)=\cos x, f^{\prime \prime}(x)=-\sin x, f^{\prime \prime \prime}(x)=-\cos x, \cdots ; \\
& f(0)=0, f^{\prime}(0)=1, f^{\prime \prime}(0)=0, f^{\prime \prime \prime}(0)=-1, f^{(4)}(0)=0, \cdots ;
\end{aligned}
$$

Taylor Polynomials of $f(x)=\sin x$

$$
\begin{aligned}
& P_{0}(x)=0, \\
& P_{1}(x)=P_{2}(x)=x, \\
& P_{3}(x)=P_{4}(x)=x-\frac{x^{3}}{3!}, \\
& P_{5}(x)=P_{6}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}, \\
& P_{7}(x)=P_{8}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!},
\end{aligned}
$$

## Remainder Term

## Remainder Term

Define the nth remainder by $R_{n}(x)=f(x)-P_{n}(x)$; that is $f(x)=P_{n}(x)+R_{n}(x)$

## Taylor's Theorem

Lagrange Formula for the Remainder

## Remainder Term

## Remainder Term

Define the $n$th remainder by $R_{n}(x)=f(x)-P_{n}(x)$;
that is $f(x)=P_{n}(x)+R_{n}(x)$. Then

## Taylor's Theorem

## Lagrange Formula for the Remainder

## Remainder Term

## Remainder Term

Define the $n$th remainder by $R_{n}(x)=f(x)-P_{n}(x)$; that is $f(x)=P_{n}(x)+R_{n}(x)$.


## Taylor's Theorem

## Lagrange Formula for the Remainder

## Remainder Term

## Remainder Term

Define the $n$th remainder by $R_{n}(x)=f(x)-P_{n}(x)$; that is $f(x)=P_{n}(x)+R_{n}(x)$. Then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} P_{n}(x)=f(x) \quad \text { if and only if } \tag{n}
\end{equation*}
$$

## Taylor's Theorem

## Lagrange Formula for the Remainder

## Remainder Term

## Remainder Term

Define the $n$th remainder by $R_{n}(x)=f(x)-P_{n}(x)$; that is $f(x)=P_{n}(x)+R_{n}(x)$. Then

$$
\lim _{n \rightarrow \infty} P_{n}(x)=f(x) \quad \text { if and only if } \quad \lim _{n \rightarrow \infty} R_{n}(x)=0
$$

## Taylor's Theorem

contains 0

## Lagrange Formula for the Remainder

## Remainder Term

## Remainder Term

Define the $n$th remainder by $R_{n}(x)=f(x)-P_{n}(x)$; that is $f(x)=P_{n}(x)+R_{n}(x)$. Then

$$
\lim _{n \rightarrow \infty} P_{n}(x)=f(x) \quad \text { if and only if } \quad \lim _{n \rightarrow \infty} R_{n}(x)=0
$$

## Taylor's Theorem

If $f$ has $n+1$ continuous derivatives on an open interval / that contains 0 , then for each $x \in I$

## Lagrange Formula for the Remainder

## Remainder Term

## Remainder Term

Define the $n$th remainder by $R_{n}(x)=f(x)-P_{n}(x)$; that is $f(x)=P_{n}(x)+R_{n}(x)$. Then

$$
\lim _{n \rightarrow \infty} P_{n}(x)=f(x) \quad \text { if and only if } \quad \lim _{n \rightarrow \infty} R_{n}(x)=0
$$

## Taylor's Theorem

If $f$ has $n+1$ continuous derivatives on an open interval $/$ that contains 0 , then for each $x \in 1$,

## Lagrange Formula for the Remainder

## Remainder Term

## Remainder Term

Define the $n$th remainder by $R_{n}(x)=f(x)-P_{n}(x)$; that is $f(x)=P_{n}(x)+R_{n}(x)$. Then

$$
\lim _{n \rightarrow \infty} P_{n}(x)=f(x) \quad \text { if and only if } \quad \lim _{n \rightarrow \infty} R_{n}(x)=0
$$

## Taylor's Theorem

If $f$ has $n+1$ continuous derivatives on an open interval $/$ that contains 0 , then for each $x \in I$,
$f^{(n+1)}(t)(x-t)^{n} d t$.
Lagrange Formula for the Remainder
For some number c between 0 and

## Remainder Term

## Remainder Term

Define the $n$th remainder by $R_{n}(x)=f(x)-P_{n}(x)$; that is $f(x)=P_{n}(x)+R_{n}(x)$. Then

$$
\lim _{n \rightarrow \infty} P_{n}(x)=f(x) \quad \text { if and only if } \quad \lim _{n \rightarrow \infty} R_{n}(x)=0
$$

## Taylor's Theorem

If $f$ has $n+1$ continuous derivatives on an open interval $/$ that contains 0 , then for each $x \in I$,

$$
R_{n}(x)=\frac{1}{n!} \int_{0}^{x} f^{(n+1)}(t)(x-t)^{n} d t
$$

## Lagrange Formula for the Remainder

For some number $c$ between 0 and $x$,

## Remainder Term

## Remainder Term

Define the $n$th remainder by $R_{n}(x)=f(x)-P_{n}(x)$; that is $f(x)=P_{n}(x)+R_{n}(x)$. Then

$$
\lim _{n \rightarrow \infty} P_{n}(x)=f(x) \quad \text { if and only if } \quad \lim _{n \rightarrow \infty} R_{n}(x)=0
$$

## Taylor's Theorem

If $f$ has $n+1$ continuous derivatives on an open interval $/$ that contains 0 , then for each $x \in I$,

$$
R_{n}(x)=\frac{1}{n!} \int_{0}^{x} f^{(n+1)}(t)(x-t)^{n} d t
$$

Lagrange Formula for the Remainder
For some number $c$ between 0 and $x$,


## Remainder Term

## Remainder Term

Define the $n$th remainder by $R_{n}(x)=f(x)-P_{n}(x)$; that is $f(x)=P_{n}(x)+R_{n}(x)$. Then

$$
\lim _{n \rightarrow \infty} P_{n}(x)=f(x) \quad \text { if and only if } \quad \lim _{n \rightarrow \infty} R_{n}(x)=0
$$

## Taylor's Theorem

If $f$ has $n+1$ continuous derivatives on an open interval $/$ that contains 0 , then for each $x \in I$,

$$
R_{n}(x)=\frac{1}{n!} \int_{0}^{x} f^{(n+1)}(t)(x-t)^{n} d t
$$

Lagrange Formula for the Remainder
For some number $c$ between 0 and $x$,

$$
R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}
$$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

Taylor Polynomials of the Exponential $f(x)=e^{x}$

## Remainder Term

## Proof.

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
f(x)=e^{x},
$$

## Remainder Term

## Proof.

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
f(x)=e^{x}, \quad P_{n}(x)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots+\frac{x^{n}}{n!} .
$$

## Remainder Term



Proof.
Let $J$ be the interval that joins 0 to $x$ and let $M=$ max $e$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
f(x)=e^{x}, \quad P_{n}(x)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots+\frac{x^{n}}{n!} .
$$

## Remainder Term

For each real $x, R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.

## Proof.

Let $J$ be the interval that joins 0 to $x$ and let $M=\max e^{t}$ Note that $f^{(n+1)}(t)=e^{t}$ for all $n$, then $\max \left|f^{(n+1)}(t)\right|=M$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
f(x)=e^{x}, \quad P_{n}(x)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots+\frac{x^{n}}{n!} .
$$

## Remainder Term

For each real $x, R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.

## Proof.

Let $J$ be the interval that joins 0 to $x$ and let $M=\max e^{t}$. Note that $f^{(n+1)}(t)=e^{t}$ for all $n$, then $\max \left|f^{(n+1)}(t)\right|=M$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
f(x)=e^{x}, \quad P_{n}(x)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots+\frac{x^{n}}{n!} .
$$

## Remainder Term

For each real $x, R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.

## Proof.

Let $J$ be the interval that joins 0 to $x$ and let $M=\max _{t \in J} e^{t}$. Note that $f^{(n+1)}(t)=e^{t}$ for all $n$, then $\max _{t \in J}\left|f^{(n+1)}(t)\right|=M$.


## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
f(x)=e^{x}, \quad P_{n}(x)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots+\frac{x^{n}}{n!} .
$$

## Remainder Term

For each real $x, R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.

## Proof.

Let $J$ be the interval that joins 0 to $x$ and let $M=\max _{t \in J} e^{t}$. Note that $f^{(n+1)}(t)=e^{t}$ for all $n$, then $\max _{t \in J}\left|f^{(n+1)}(t)\right|=\mathrm{M}$.

$$
\left|R_{n}(x)\right| \leq M \frac{|x|^{n+1}}{(n+1)!}
$$

## Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

Taylor Polynomials of the Exponential $f(x)=e^{x}$

$$
f(x)=e^{x}, \quad P_{n}(x)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots+\frac{x^{n}}{n!} .
$$

## Remainder Term

For each real $x, R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.

## Proof.

Let $J$ be the interval that joins 0 to $x$ and let $M=\max _{t \in J} e^{t}$. Note that $f^{(n+1)}(t)=e^{t}$ for all $n$, then $\max _{t \in J}\left|f^{(n+1)}(t)\right|=M$.

$$
\left|R_{n}(x)\right| \leq M \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty
$$

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ;
$$



Taylor Polynomials of the Sine $f(x)=\sin x$

Remainder Term
$\qquad$

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

Taylor Polynomials of the Sine $f(x)=\sin x$

Remainder Term
$\qquad$

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

$$
\left|R_{n}(x)\right| \leq\left(\max _{t \in J}\left|f^{(n+1)}(t)\right|\right) \frac{|x|^{n+1}}{(n+1)!}, \quad J=[0, x] \text { or }[x, 0] .
$$

Taylor Polynomials of the Sine $f(x)=\sin x$

Remainder Term
$\qquad$

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

$$
\left|R_{n}(x)\right| \leq\left(\max _{t \in J}\left|f^{(n+1)}(t)\right|\right) \frac{|x|^{n+1}}{(n+1)!}, \quad J=[0, x] \text { or }[x, 0] .
$$

Taylor Polynomials of the Sine $f(x)=\sin x$

$$
\begin{equation*}
f(x)=\sin x, \tag{7}
\end{equation*}
$$

$\qquad$

Remainder Term
For each rea
$R_{n}(x) \rightarrow 0$ as $n$

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

$$
\left|R_{n}(x)\right| \leq\left(\max _{t \in J}\left|f^{(n+1)}(t)\right|\right) \frac{|x|^{n+1}}{(n+1)!}, \quad J=[0, x] \text { or }[x, 0] .
$$

Taylor Polynomials of the Sine $f(x)=\sin x$

$$
f(x)=\sin x, \quad P_{7}(x)=P_{8}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}, \text { and so on. }
$$

Remainder Term
For each real $x, R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

$$
\left|R_{n}(x)\right| \leq\left(\max _{t \in J}\left|f^{(n+1)}(t)\right|\right) \frac{|x|^{n+1}}{(n+1)!}, \quad J=[0, x] \text { or }[x, 0] .
$$

Taylor Polynomials of the Sine $f(x)=\sin x$

$$
f(x)=\sin x, \quad P_{7}(x)=P_{8}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}, \text { and so on. }
$$

## Remainder Term

For each real $x, R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.


## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

$$
\left|R_{n}(x)\right| \leq\left(\max _{t \in J}\left|f^{(n+1)}(t)\right|\right) \frac{|x|^{n+1}}{(n+1)!}, \quad J=[0, x] \text { or }[x, 0] .
$$

Taylor Polynomials of the Sine $f(x)=\sin x$

$$
f(x)=\sin x, \quad P_{7}(x)=P_{8}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}, \text { and so on. }
$$

## Remainder Term

For each real $x, R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.
$\forall k, f^{(k)}(t)= \pm \cos t$ or $\pm \sin t$,

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

$$
\left|R_{n}(x)\right| \leq\left(\max _{t \in J}\left|f^{(n+1)}(t)\right|\right) \frac{|x|^{n+1}}{(n+1)!}, \quad J=[0, x] \text { or }[x, 0] .
$$

Taylor Polynomials of the Sine $f(x)=\sin x$

$$
f(x)=\sin x, \quad P_{7}(x)=P_{8}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}, \text { and so on. }
$$

## Remainder Term

For each real $x, R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.
$\forall k, f^{(k)}(t)= \pm \cos t$ or $\pm \sin t$, then $\max _{t \in J}\left|f^{(n+1)}(t)\right| \leq 1$.

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

$$
\left|R_{n}(x)\right| \leq\left(\max _{t \in J}\left|f^{(n+1)}(t)\right|\right) \frac{|x|^{n+1}}{(n+1)!}, \quad J=[0, x] \text { or }[x, 0] .
$$

Taylor Polynomials of the Sine $f(x)=\sin x$

$$
f(x)=\sin x, \quad P_{7}(x)=P_{8}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}, \text { and so on. }
$$

## Remainder Term

For each real $x, R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.
$\forall k, f^{(k)}(t)= \pm \cos t$ or $\pm \sin t$, then $\max _{t \in J}\left|f^{(n+1)}(t)\right| \leq 1$.

$$
\left|R_{n}(x)\right| \leq \frac{|x|^{n+1}}{(n+1)!}
$$

## Taylor Polynomials of the Sine $f(x)=\sin x$

$$
P_{n}(x)=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n} ; \quad R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} .
$$

$$
\left|R_{n}(x)\right| \leq\left(\max _{t \in J}\left|f^{(n+1)}(t)\right|\right) \frac{|x|^{n+1}}{(n+1)!}, \quad J=[0, x] \text { or }[x, 0] .
$$

Taylor Polynomials of the Sine $f(x)=\sin x$

$$
f(x)=\sin x, \quad P_{7}(x)=P_{8}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}, \text { and so on. }
$$

## Remainder Term

For each real $x, R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.
$\forall k, f^{(k)}(t)= \pm \cos t$ or $\pm \sin t$, then $\max _{t \in J}\left|f^{(n+1)}(t)\right| \leq 1$.

$$
\left|R_{n}(x)\right| \leq \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty .
$$

## Taylor Series

Taylor Polynomial and the Remainder
If $f(x)$ is infinitely differentiable on interval / containing 0 , then

## Taylor Series

## Taylor Series

Taylor Polynomial and the Remainder
If $f(x)$ is infinitely differentiable on interval / containing 0 , then

$$
f(x)=P_{n}(x)+R_{n}(x), \quad \forall x \in I ;
$$

## Taylor Series

## Taylor Series

Taylor Polynomial and the Remainder
If $f(x)$ is infinitely differentiable on interval / containing 0 , then

$$
f(x)=P_{n}(x)+R_{n}(x), \quad \forall x \in I ;
$$

$$
P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}
$$



## Taylor Series

$\qquad$

## Taylor Series

## Taylor Polynomial and the Remainder

If $f(x)$ is infinitely differentiable on interval / containing 0 , then

$$
\begin{gathered}
f(x)=P_{n}(x)+R_{n}(x), \quad \forall x \in I ; \\
P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}, \\
R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \text { or } R_{n}(x)=\frac{1}{n!} \int_{0}^{x} f^{(n+1)}(t)(x-t)^{n} d t .
\end{gathered}
$$

## Taylor Series

If $R_{n}(x) \rightarrow 0$ as $r$

## Taylor Series

## Taylor Polynomial and the Remainder

If $f(x)$ is infinitely differentiable on interval / containing 0 , then

$$
\begin{gathered}
f(x)=P_{n}(x)+R_{n}(x), \quad \forall x \in I ; \\
P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}, \\
R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \text { or } R_{n}(x)=\frac{1}{n!} \int_{0}^{x} f^{(n+1)}(t)(x-t)^{n} d t
\end{gathered}
$$

## Taylor Series

If $R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$,

In this case, $f(x)$ can be expanded as a Taylor series in $x$ and write

## Taylor Series

## Taylor Polynomial and the Remainder

If $f(x)$ is infinitely differentiable on interval / containing 0 , then

$$
\begin{gathered}
f(x)=P_{n}(x)+R_{n}(x), \quad \forall x \in I ; \\
P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}, \\
R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \text { or } R_{n}(x)=\frac{1}{n!} \int_{0}^{x} f^{(n+1)}(t)(x-t)^{n} d t .
\end{gathered}
$$

## Taylor Series

If $R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$, then $P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k} \rightarrow f(x)$.
In this case, $f(x)$ can be expanded as a Taylor series in $x$ and write

## Taylor Series

## Taylor Polynomial and the Remainder

If $f(x)$ is infinitely differentiable on interval / containing 0 , then

$$
\begin{gathered}
f(x)=P_{n}(x)+R_{n}(x), \quad \forall x \in I ; \\
P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}=f(0)+f^{\prime}(0) x+\cdots+\frac{f^{(n)}(0)}{n!} x^{n}, \\
R_{n}(x)=\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \text { or } R_{n}(x)=\frac{1}{n!} \int_{0}^{x} f^{(n+1)}(t)(x-t)^{n} d t
\end{gathered}
$$

## Taylor Series

If $R_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$, then $P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k} \rightarrow f(x)$.
In this case, $f(x)$ can be expanded as a Taylor series in $x$ and write

$$
f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}
$$

## Taylor Series of the Exponential $f(x)=e^{x}$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Exponential $f(x)=e^{x}$

Numbere

## Taylor Series of the Exponential $f(x)=e^{x}$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Exponential $f(x)=e^{x}$

$$
e^{x}=\sum_{k=0}^{\infty} \frac{x^{k}}{k!}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots
$$

Numbere

## Taylor Series of the Exponential $f(x)=e^{x}$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Exponential $f(x)=e^{x}$

$$
e^{x}=\sum_{k=0}^{\infty} \frac{x^{k}}{k!}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots \quad \text { for all real } x
$$

Numbere


## Taylor Series of the Exponential $f(x)=e^{x}$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Exponential $f(x)=e^{x}$

$$
e^{x}=\sum_{k=0}^{\infty} \frac{x^{k}}{k!}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots \quad \text { for all real } x
$$

Numbere

$$
e=\sum_{k=0}^{\infty} \frac{1}{k!}=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots
$$

## Taylor Series of the Sine $f(x)=\sin x$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Sine $f(x)=\sin x$


Number $\sin 1$

## Taylor Series of the Sine $f(x)=\sin x$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Sine $f(x)=\sin x$

$$
\sin x=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2 k+1)!} x^{2 k+1}=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\cdots
$$

Number $\sin 1$

## Taylor Series of the Sine $f(x)=\sin x$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Sine $f(x)=\sin x$

$$
\sin x=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2 k+1)!} x^{2 k+1}=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\cdots \quad \text { for all real } x
$$

## Number $\sin 1$



## Taylor Series of the Sine $f(x)=\sin x$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Sine $f(x)=\sin x$

$$
\sin x=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2 k+1)!} x^{2 k+1}=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\cdots \quad \text { for all real } x
$$

Number $\sin 1$

$$
\sin 1=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2 k+1)!}=1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\cdots
$$

## Taylor Series of the Cosine $f(x)=\cos x$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Cosine $f(x)=\cos x$


Number $\cos 1$

## Taylor Series of the Cosine $f(x)=\cos x$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Cosine $f(x)=\cos x$

$$
\cos x=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2 k)!} x^{2 k}=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}+\cdots
$$

Number $\cos 1$

## Taylor Series of the Cosine $f(x)=\cos x$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Cosine $f(x)=\cos x$

$$
\cos x=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2 k)!} x^{2 k}=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}+\cdots \quad \text { for all real } x
$$

Number $\cos 1$

## Taylor Series of the Cosine $f(x)=\cos x$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Cosine $f(x)=\cos x$

$$
\cos x=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2 k)!} x^{2 k}=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}+\cdots \quad \text { for all real } x
$$

Number $\cos 1$

$$
\cos 1=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2 k)!}=1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\cdots
$$

## Taylor Series of the Logarithm $f(x)=\ln (1+x)$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Logarithm $f(x)=\ln (1+x)$

Number $\ln 2$

## Taylor Series of the Logarithm $f(x)=\ln (1+x)$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Logarithm $f(x)=\ln (1+x)$

$$
\ln (1+x)=\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^{k}=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{4}+\cdots
$$

Number $\ln 2$

## Taylor Series of the Logarithm $f(x)=\ln (1+x)$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Logarithm $f(x)=\ln (1+x)$

$$
\ln (1+x)=\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^{k}=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{4}+\cdots \text { for }-1<x \leq 1
$$

Number $\ln 2$

## Taylor Series of the Logarithm $f(x)=\ln (1+x)$

$$
f(x)=P_{n}(x)+R_{n}(x), \quad P_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k}
$$

If $\lim _{n \rightarrow \infty} R_{n}(x) \rightarrow 0$, then $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^{k}=\lim _{n \rightarrow \infty} P_{n}(x)$.
Taylor Series of the Logarithm $f(x)=\ln (1+x)$

$$
\ln (1+x)=\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^{k}=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{4}+\cdots \text { for }-1<x \leq 1
$$

Number $\ln 2$

$$
\ln 2=\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots
$$

## Outline

- Taylor Polynomials
- Taylor Polynomials
- Remainder Term
- Taylor Series
- Taylor Series
- Numerical Calculations

