Lecture 25

Section 11.5 Taylor Polynomials in x; Taylor Series in x

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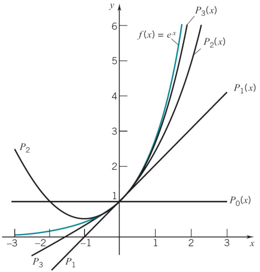

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$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n;$$

P_n is the polynomial that has the same value as f at 0 and the same first n derivatives:

$$P_n(0) = f(0), P_n'(0) = f'(0), P_n''(0) = f''(0), \cdots, P_n^{(n)}(0) = f^{(n)}(0).$$

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P_n provides the best local approximation of $f(x)$ near 0 by a polynomial of degree $\le n$.

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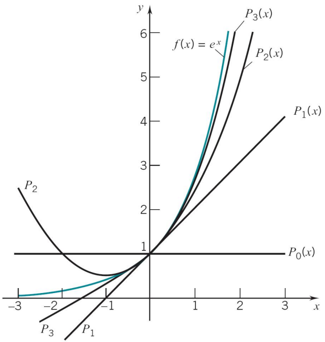

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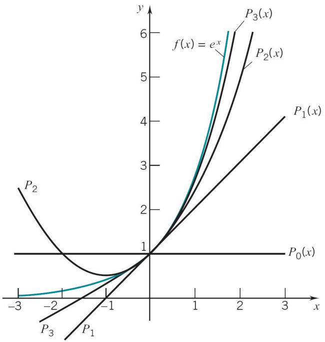

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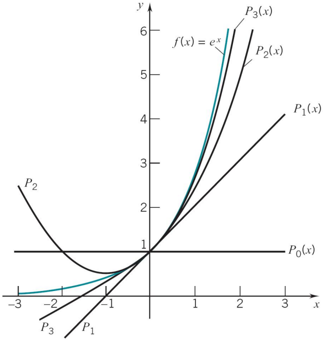

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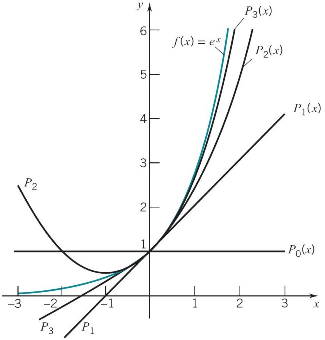

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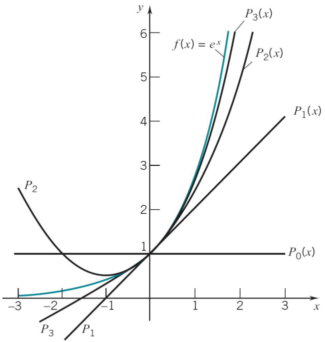

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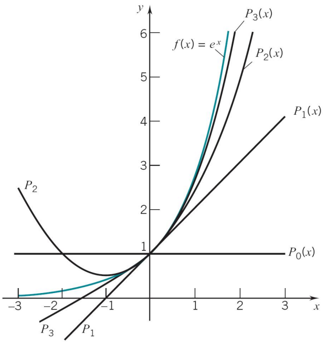

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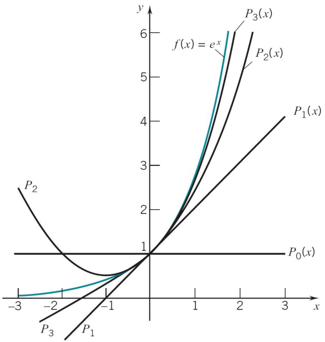

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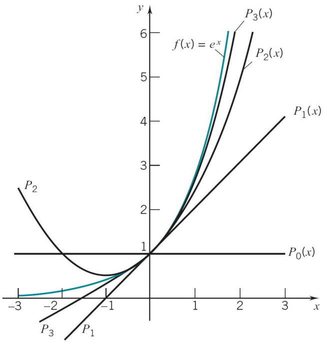

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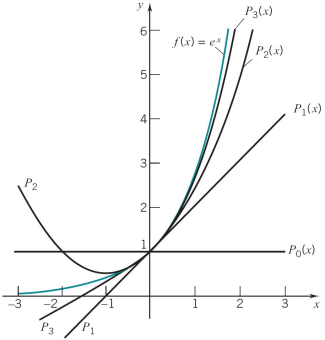

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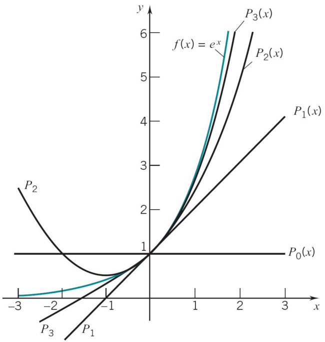

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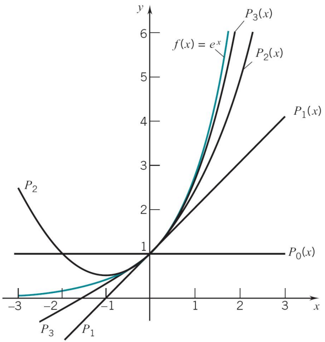

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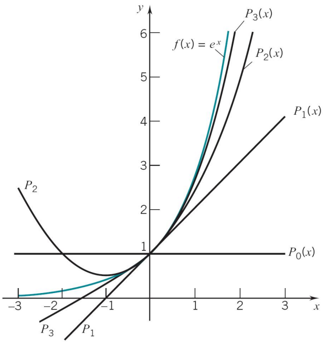

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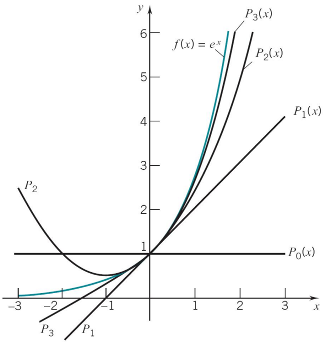

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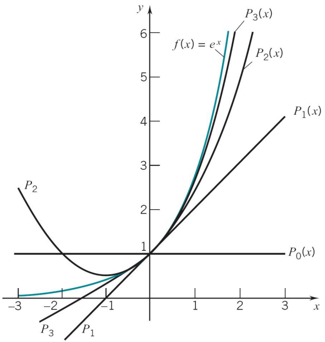

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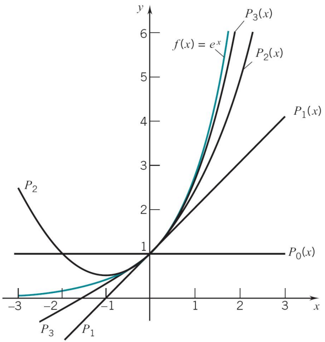

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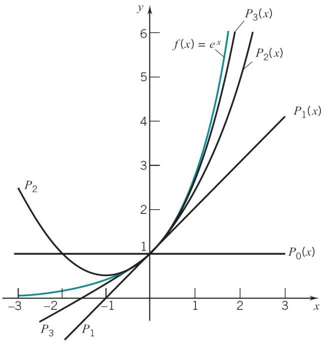

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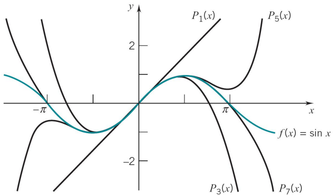

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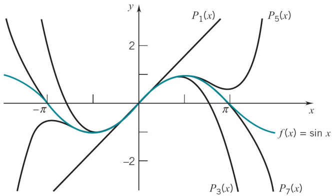

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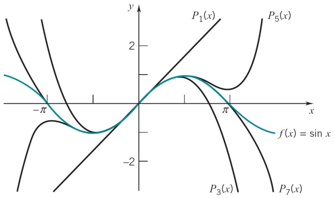

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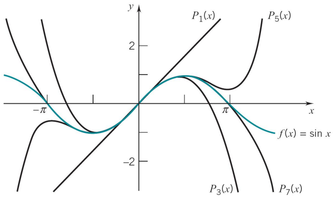

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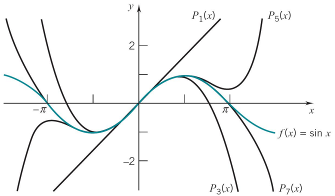

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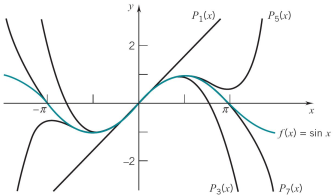

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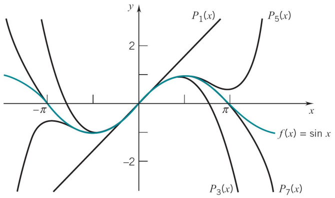

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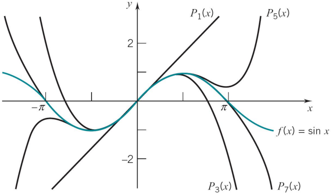

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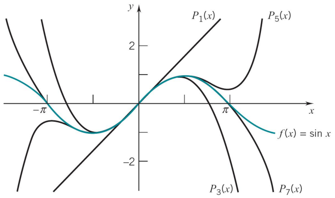

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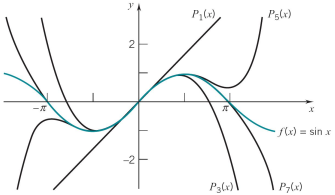

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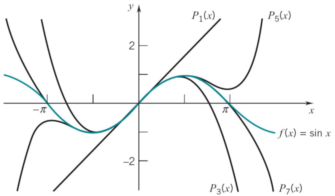

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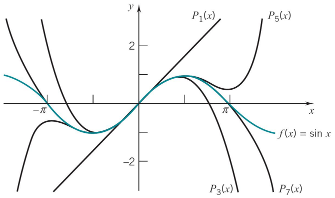

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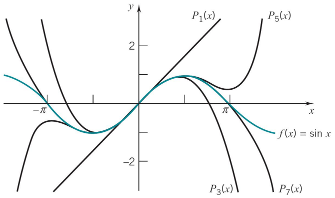

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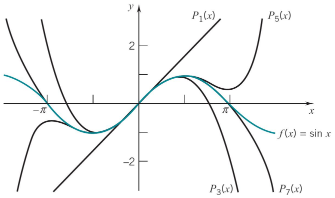

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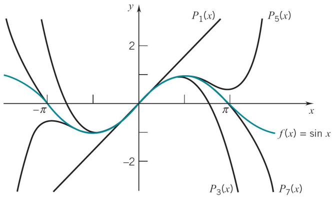
$$f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x, \cdots;$$
$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f^{(4)}(0) = 0, \cdots;$$



Taylor Polynomials of $f(x) = \sin x$

$$P_0(x) = 0,$$
$$P_1(x) = P_2(x) = x,$$
$$P_3(x) = P_4(x) = x - \frac{x^3}{3!},$$
$$P_5(x) = P_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!},$$
$$P_7(x) = P_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!},$$

Remainder Term

Remainder Term

Define the *nth remainder* by $R_n(x) = f(x) - P_n(x)$; that is $f(x) = P_n(x) + R_n(x)$. Then

$$\lim_{n\to\infty} P_n(x) = f(x) \quad \text{if and only if} \quad \lim_{n\to\infty} R_n(x) = 0$$

Taylor's Theorem

If f has $n+1$ continuous derivatives on an open interval I that contains 0, then for each $x \in I$,

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n \, dt.$$

Lagrange Formula for the Remainder

For some number c between 0 and x,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

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For each real x, $R_n(x) \to 0$ as $n \to \infty$.

Proof.

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For each real x, $R_n(x) \to 0$ as $n \to \infty$.

$\forall k$, $f^{(k)}(t) = \pm\cos t$ or $\pm\sin t$, then $\max_{t\in J} |f^{(n+1)}(t)| \le 1.$

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \to 0 \quad \text{as} \quad n \to \infty.$$

Taylor Polynomials of the Sine $f(x) = \sin x$

$$P_n(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n; \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

$$|R_n(x)| \le \left(\max_{t \in J} |f^{(n+1)}(t)|\right) \frac{|x|^{n+1}}{(n+1)!}, \quad J = [0, x] \text{ or } [x, 0].$$

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Taylor Series

Taylor Polynomial and the Remainder

If $f(x)$ is infinitely differentiable on interval I containing 0, then

$$f(x) = P_n(x) + R_n(x), \quad \forall x \in I;$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n,$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \quad \text{or} \quad R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt.$$

Taylor Series

If $R_n(x) \to 0$ as $n \to \infty$, then $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k \to f(x)$.

In this case, $f(x)$ can be expanded as a Taylor series in x and write

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Taylor Series of the Exponential $f(x) = e^x$

$$f(x) = P_n(x) + R_n(x), \quad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

If $\lim_{n\to\infty} R_n(x) \to 0$, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n\to\infty} P_n(x)$.

Taylor Series of the Exponential $f(x) = e^x$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{for all real } x$$

Number e

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

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Number $\sin 1$

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If $\lim_{n\to\infty} R_n(x) \to 0$, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n\to\infty} P_n(x)$.

Taylor Series of the Logarithm $f(x) = \ln(1+x)$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } -1 < x \leq 1$$

Number $\ln 2$

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Outline

- Taylor Polynomials
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  - Remainder Term
- Taylor Series
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  - Numerical Calculations