Lecture 25 Section 11.5 Taylor Polynomials in x; Taylor Series in x

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1 Taylor Polynomials

1.1 Taylor Polynomials

Taylor Polynomials
Taylor Polynomials

The *nth Taylor polynomial* at 0 for a function f is

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n;$$

P_n is the polynomial that has the *same value* as f at 0 and the *same first n derivatives*:

$$P_n(0) = f(0), P_n'(0) = f'(0), P_n''(0) = f''(0), \cdots, P_n^{(n)}(0) = f^{(n)}(0).$$

Best Approximation

P_n provides the *best local approximation* of $f(x)$ near 0 by a polynomial of degree $\leq n$.

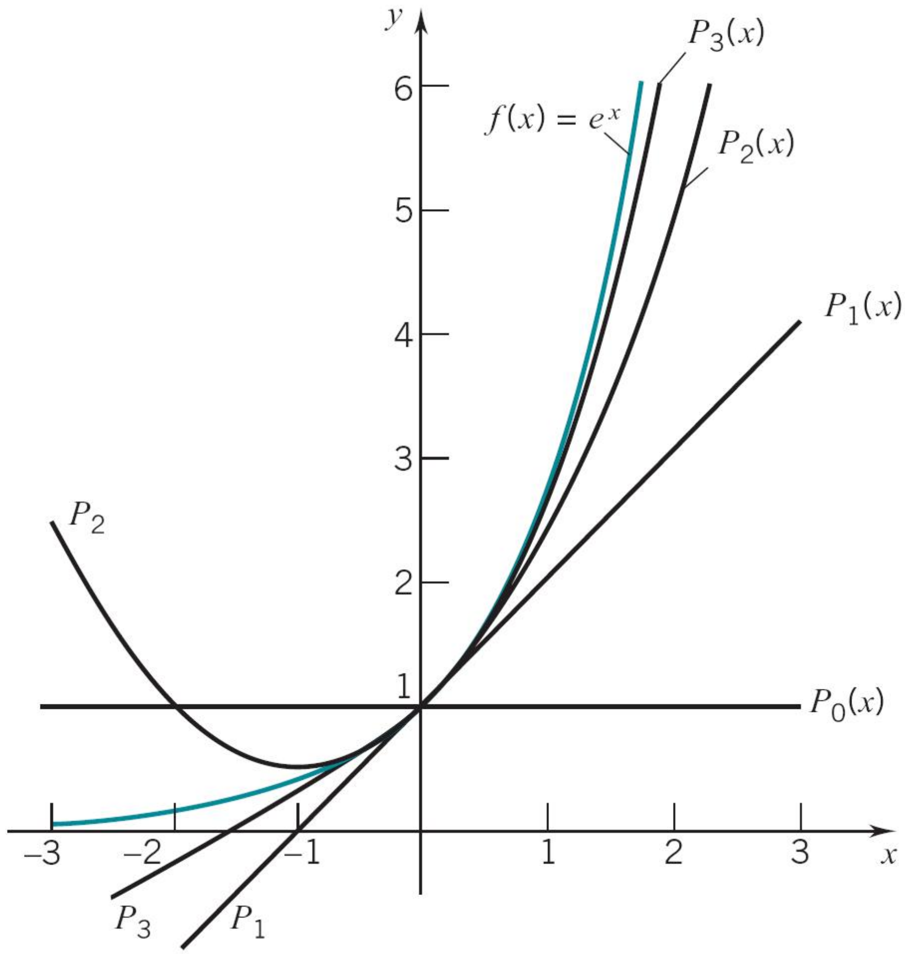
$$\begin{aligned} P_0(x) &= f(0), \\ P_1(x) &= f(0) + f'(0)x, \\ P_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2. \end{aligned}$$

Taylor Polynomials of the Exponential $f(x) = e^x$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n;$$

$$f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \quad \cdots, \quad f^{(n)}(x) = e^x;$$

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad \cdots, \quad f^{(n)}(0) = 1.$$



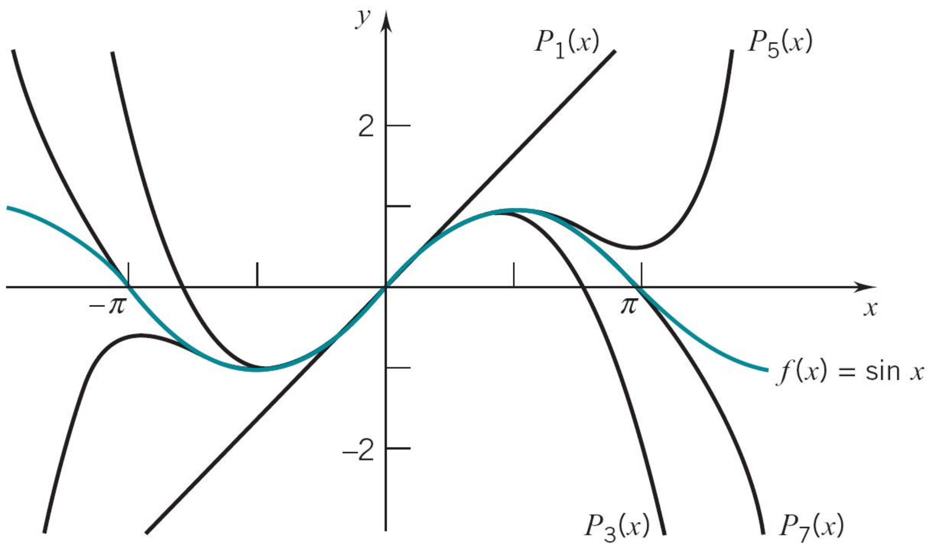
Taylor Polynomials of $f(x) = e^x$

$$P_0(x) = 1,$$
$$P_1(x) = 1 + x,$$
$$P_2(x) = 1 + x + \frac{x^2}{2!},$$
$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!},$$
$$\vdots$$
$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Taylor Polynomials of the Sine $f(x) = \sin x$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n;$$

$$f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x,\ \cdots;$$



Taylor Polynomials of $f(x) = \sin x$

$$\begin{aligned}
P_0(x) &= 0, \\
P_1(x) &= P_2(x) = x, \\
P_3(x) &= P_4(x) = x - \frac{x^3}{3!}, \\
P_5(x) &= P_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}, \\
P_7(x) &= P_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!},
\end{aligned}$$

1.2 Remainder Term

Remainder Term
Remainder Term

Define the *nth remainder* by $R_n(x) = f(x) - P_n(x)$; that is $f(x) = P_n(x) + R_n(x)$. Then

$$\lim_{n\to\infty} P_n(x) = f(x) \quad \text{if and only if} \quad \lim_{n\to\infty} R_n(x) = 0$$

Taylor's Theorem

If f has $n+1$ *continuous derivatives* on an open interval I that contains 0, then for *each* $x \in I$,

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt.$$

Lagrange Formula for the Remainder

For some number c between 0 and x,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Taylor Polynomials of the Exponential $f(x) = e^x$

$$P_n(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n; \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Taylor Polynomials of the Exponential $f(x) = e^x$

$$f(x) = e^x, \quad P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Remainder Term

For each real x, $R_n(x) \to 0$ as $n \to \infty$.

Proof.

Let J be the interval that joins 0 to x and let $M = \max_{t\in J} e^t$. Note that $f^{(n+1)}(t) = e^t$ for all n, then $\max_{t\in J}|f^{(n+1)}(t)| = \text{M}$.

$$|R_n(x)| \le M\frac{|x|^{n+1}}{(n+1)!} \to 0 \quad \text{as} \quad n \to \infty.$$

Taylor Polynomials of the Sine $f(x) = \sin x$

$$P_n(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n; \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

$$|R_n(x)| \le \left(\max_{t\in J}|f^{(n+1)}(t)|\right)\frac{|x|^{n+1}}{(n+1)!}, \quad J = [0,x] \text{ or } [x,0].$$

Taylor Polynomials of the Sine $f(x) = \sin x$

$$f(x) = \sin x, \quad P_7(x) = P_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}, \text{ and so on.}$$

Remainder Term

For each real x, $R_n(x) \to 0$ as $n \to \infty$.

$\forall k$, $f^{(k)}(t) = \pm\cos t$ or $\pm\sin t$, then $\max_{t\in J}|f^{(n+1)}(t)| \le 1$.

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \to 0 \quad \text{as} \quad n \to \infty.$$

2 Taylor Series

2.1 Taylor Series

Taylor Series

Taylor Polynomial and the Remainder

If $f(x)$ is infinitely differentiable on interval I containing 0, then

$$f(x) = P_n(x) + R_n(x), \quad \forall x \in I;$$

$$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n,$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \text{ or } R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt.$$

Taylor Series

If $R_n(x) \to 0$ as $n \to \infty$, then $P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k \to f(x)$. [1ex] In this case, $f(x)$ can be expanded as a *Taylor series* in x and write

$$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k.$$

Taylor Series of the Exponential $f(x) = e^x$

$$f(x) = P_n(x) + R_n(x), \quad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

If $\lim_{n\to\infty} R_n(x) \to 0$, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n\to\infty} P_n(x)$.

Taylor Series of the Exponential $f(x) = e^x$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{for all real } x$$

Number e

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Taylor Series of the Sine $f(x) = \sin x$

$$f(x) = P_n(x) + R_n(x), \quad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

If $\lim_{n\to\infty} R_n(x) \to 0$, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n\to\infty} P_n(x)$.

Taylor Series of the Sine $f(x) = \sin x$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{for all real } x$$

Number $\sin 1$

$$\sin 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$$

Taylor Series of the Cosine $f(x) = \cos x$

$$f(x) = P_n(x) + R_n(x), \quad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

If $\lim_{n\to\infty} R_n(x) \to 0$, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n\to\infty} P_n(x)$.

Taylor Series of the Cosine $f(x) = \cos x$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{for all real } x$$

Number $\cos 1$

$$\cos 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots$$

Taylor Series of the Logarithm $f(x) = \ln(1+x)$

$$f(x) = P_n(x) + R_n(x), \quad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

If $\lim_{n\to\infty} R_n(x) \to 0$, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n\to\infty} P_n(x)$.

Taylor Series of the Logarithm $f(x) = \ln(1+x)$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } -1 < x \leq 1$$

Number $\ln 2$

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

2.2 Numerical Calculations

Outline

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