Lecture 26 Section 11.6 Taylor Polynomials and Taylor Series in $x - a$

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1 Taylor Polynomials in $x - a$

1.1 Taylor Polynomials in $x - a$

Taylor Polynomials in Powers of $x - a$
Taylor Polynomials in Powers of $x - a$

The *nth Taylor polynomial* in $x - a$ for a function f is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n;$$

P_n is the polynomial that has the *same value* as f at a and the *same first n derivatives*:

$$P_n(a) = f(a), P_n'(a) = f'(a), P_n''(a) = f''(a), \cdots, P_n^{(n)}(a) = f^{(n)}(a).$$

Best Approximation

P_n provides the *best local approximation* of $f(x)$ near a by a polynomial of degree $\leq n$.

$$\begin{aligned} P_0(x) &= f(a), \\ P_1(x) &= f(a) + f'(a)(x-a), \\ P_2(x) &= f(a) + f'(a)x + \frac{f''(a)}{2!}(x-a)^2. \end{aligned}$$

Taylor's Theorem and Remainder Term

Taylor's Theorem

If f has $n+1$ *continuous derivatives* on an open interval I that contains a, then for *each* $x \in I$,

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x); \\ R_n(x) &= \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt. \end{aligned}$$

Lagrange Formula for the Remainder

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where c is some number between a and x.

Estimate for the Remainder Term

$$|R_n(x)| \leq \left(\max_{t \in J} |f^{(n+1)}(t)|\right) \frac{|x-a|^{n+1}}{(n+1)!}, \quad J = [a, x] \text{ or } [x, a].$$

1.2 Taylor Series in $x - a$

Taylor Series in $x - a$
Taylor Polynomial and the Remainder
If $f(x)$ is infinitely differentiable on interval I containing a, then

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x);$$

$$|R_n(x)| \le \left(\max_{t\in J} |f^{(n+1)}(t)|\right) \frac{|x-a|^{n+1}}{(n+1)!}, \quad J = [a, x] \text{ or } [x, a].$$

Taylor Series in $x - a$
If $R_n(x) \to 0$ as $n \to \infty$, then $P_n(x) \to f(x)$,

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

Sigma Notation

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Taylor Polynomials and Taylor Series in $x - a$ of $f(x) = e^x$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n;$$

$$f^{(k)}(x) = e^x, \quad f^{(k)}(a) = e^a, \quad \forall k = 0, 1, 2, \cdots$$

Taylor Polynomials in $x - a$ of the Exponential $f(x) = e^x$

$$P_n(x) = e^a + e^a(x-a) + \frac{e^a}{2!}(x-a)^2 + \cdots + \frac{e^a}{n!}(x-a)^n.$$

Taylor Series in $x - a$ of the Exponential $f(x) = e^x$

$$e^x = e^a \sum_{k=0}^{\infty} \frac{1}{k!}(x-a)^k = e^a + e^a(x-a) + \cdots + \frac{e^a}{n!}(x-a)^n + \cdots, \quad \forall x.$$

Expansion of e^x in $x - a$

Taylor Series in $x - a$ by Translation
Another way to *expand $f(x)$ in in powers of $x-a$* is to *expand $f(t+a)$ in powers of t* and then set $t = x - a$.

Taylor Series in $x - a$ of the Exponential $f(x) = e^x$

$$e^x = e^a + e^a(x-a) + \frac{e^a}{2!}(x-a)^2 + \cdots + \frac{e^a}{n!}(x-a)^n + \cdots, \quad \forall x.$$

1. Expand e^{t+a} in powers of t $\Rightarrow$

$$e^{t+a} = e^a e^t = e^a \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^a \sum_{k=0}^{\infty} \frac{1}{k!} t^k.$$

2. Set $t = x - a$ $\Rightarrow$

$$e^x = e^a \sum_{k=0}^{\infty} \frac{1}{k!}(x-a)^k, \quad \text{for all real x.}$$

1.3 Powers in $x - a$ by Translation

Expansion of $f(x)$ in $x - a$ as Expansion of $f(t + a)$ in t
Taylor Series in $x - a$ by Translation
One way to *expand* $f(x)$ *in in powers of* $x - a$ is to *expand* $f(t + a)$ *in powers of* t and then set $t = x - a$. This is the approach to take when the *expansion in* t is either *known* or is readily available.

Example 1. Expand $f(x) = e^{x/2}$ in powers of $x - 3$.

1. Expand $f(t + 3)$ in powers of t $\Rightarrow$

$$f(t+3) = e^{(t+3)/2} = e^{3/2}e^{t/2} = e^{3/2}\sum_{k=0}^{\infty}\frac{(t/2)^k}{k!} = e^{3/2}\sum_{k=0}^{\infty}\frac{1}{2^k k!}t^k.$$

2. Set $t = x - 3$ $\Rightarrow$

$$f(x) = e^{x/2} = e^{3/2}\sum_{k=0}^{\infty}\frac{1}{2^k k!}(x-3)^k, \qquad \text{for all real x.}$$

Expansion of $\ln x$ in $x - a$, $a > 0$
Taylor Series in $x - a$ by Translation
One way to *expand* $f(x)$ *in in powers of* $x - a$ is to *expand* $f(t + a)$ *in powers of* t and then set $t = x - a$.

Taylor Series in $x - a$ of the Logarithm $f(x) = \ln x$

$$\ln x = \ln a + \frac{1}{a}(x-a) - \frac{1}{2a^2}(x-a)^2 + \frac{1}{3a^3}(x-a)^3 - \cdots, \quad 0 < x \le 2a.$$

1. Expand $\ln(t + a)$ in powers of t $\Rightarrow$

$$\ln(t+a) = \ln\left[a\left(1+\frac{t}{a}\right)\right] = \ln a + \ln\left(1+\frac{t}{a}\right) = \ln a + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{t}{a}\right)^k = \ln a + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{ka^k}t^k$$

2. Set $t = x - a$, $-1 < \frac{t}{a} \le 1$, $-a < t \le a$, $0 < t + a \le 2a$, $\Rightarrow$

$$\ln x = \ln a + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{ka^k}(x-a)^k, \qquad \text{for } 0 < x \le 2a.$$

Expansion of $\sin x$ in $x - \pi$

Taylor Series in $x - \pi$ of the Sine $f(x) = \sin x$

$$\sin x = -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 + \cdots, \quad \forall x.$$

1. Expand $\sin(t + \pi)$ in powers of t $\Rightarrow$

$$\sin(t+\pi) = \sin t\cos\pi + \cos t\sin\pi = -\sin t$$
$$= -\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}t^{2k+1} = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}t^{2k+1}$$

2. Set $t = x - \pi$ $\Rightarrow$

$$\sin x = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}(x-\pi)^{2k+1}, \qquad \text{for all real } x.$$

Expansion of $\cos x$ in $x - \pi$

Taylor Series in $x - \pi$ of the Cosine $f(x) = \cos x$

$$\cos x = -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 - \cdots, \quad \forall x.$$

1. Expand $\cos(t+\pi)$ in powers of t $\quad \Rightarrow$

$$\cos(t+\pi) = \cos t \cos \pi - \sin t \sin \pi = -\cos t$$
$$= -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k)!} t^{2k}$$

2. Set $t = x - \pi$ $\quad \Rightarrow$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k)!} (x-\pi)^{2k}, \quad \text{for all real } x.$$

Expansion of $\cos^2 x$ in $x - \pi$

Taylor Series in $x - \pi$ of $f(x) = \cos^2 x$

$$\cos^2 x = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1}}{(2k)!} (x-\pi)^{2k}, \quad \forall x.$$

1. Expand $\cos^2(t+\pi)$ in powers of t $\quad \Rightarrow$

$$\cos^2(t+\pi) = \frac{1}{2} + \frac{1}{2}\cos 2(t+\pi) = \frac{1}{2} + \frac{1}{2}\cos(2t + 2\pi)$$
$$= \frac{1}{2} + \frac{1}{2}\cos 2t = \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(2t)^{2k} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1}}{(2k)!} t^{2k}$$

2. Set $t = x - \pi$ $\quad \Rightarrow$

$$\cos^2 x = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1}}{(2k)!} (x-\pi)^{2k}, \quad \text{for all real } x.$$

Expansion of $(1-x)^{-1}$ in x and Related

Expansion of $(1-x)^{-1}$ in x and Geometric Series

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots, \quad |x| < 1.$$

Expand $f(x) = (1-2x)^{-1}$ in powers of $x + 2$.

1. Expand $f(t-2)$ in powers of t $\quad \Rightarrow$

$$f(t-2) = \left[1 - 2(t-2)\right]^{-1} = (5-2t)^{-1} = \frac{1}{5}\left[1 - \left(\frac{2}{5}t\right)\right]^{-1}$$
$$= \frac{1}{5}\sum_{k=0}^{\infty}\left(\frac{2}{5}t\right)^k = \frac{1}{5}\sum_{k=0}^{\infty}\left(\frac{2}{5}\right)^k t^k$$

2. Set $t = x + 2$, $-1 < \frac{2}{5}t < 1$, $-\frac{5}{2} < t < \frac{5}{2}$, $-\frac{9}{2} < t - 2 < \frac{1}{2}$,

$$f(x) = (1-2x)^{-1} = \frac{1}{5}\sum_{k=0}^{\infty}\left(\frac{2}{5}\right)^k (x+2)^k, \quad -\frac{9}{2} < x < \frac{1}{2}.$$

Expansion of $(1-x)^{-m}$ in x and Related
Expansion of $(1-x)^{-m}$ in x for $m>1$

$$(1-x)^{-m} = \frac{1}{(m-1)!}\sum_{k=0}^{\infty}(k+1)\cdots(k+m-1)x^k.$$

Expand $f(x)=(1-2x)^{-3}$ in powers of $x+2$.

1. Expand $f(t-2)$ in powers of t $\quad\Rightarrow$

$$f(t-2) = \left[1-2(t-2)\right]^{-3} = (5-2t)^{-3} = \frac{1}{5^3}\left[1-\left(\frac{2}{5}t\right)\right]^{-3}$$
$$= \frac{1}{5^3}\frac{1}{2}\sum_{k=0}^{\infty}(k+1)(k+2)\left(\frac{2}{5}t\right)^k = \sum_{k=0}^{\infty}(k+1)(k+2)\frac{2^{k-1}}{5^{k+3}}t^k$$

2. Set $t=x+2$ $\quad\Rightarrow$

$$f(x)=(1-2x)^{-3} = \sum_{k=0}^{\infty}(k+1)(k+2)\frac{2^{k-1}}{5^{k+3}}(x+2)^k$$

Outline

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