$\boxed{\text{50 points}}$ 1. Find the solution of the following initial-value problems

$$\text{a.}\quad \frac{dy}{dt} - \frac{n}{t}y = e^t t^n, \quad y(1) = 0, \quad (n \text{ integer})$$
$$\text{b.}\quad y'' + 2y' + 2y = 10\cos 2t, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution a. Solution (Integrating Factor)

$$u(t) = \exp\left(\int (-n/t)dt\right) = t^{-n}$$

$$\text{multiply ODE by } u \quad \Rightarrow \quad [t^{-n}y]' = e^t \quad \Rightarrow \quad t^{-n}y = \int e^t dt$$
$$\Rightarrow \quad y(t) = t^n(c + e^t)$$

$$y(1) = 0 = c + e \quad \Rightarrow \quad c = -e$$
$$\Rightarrow \quad y(t) = t^n(-e + e^t)$$

b. The homogeneous equation, its Characteristic Equation and roots

$$y'' + 2y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = -1 + i, \lambda_2 = -1 - i$$

The homogeneous solution is

$$y_h(t) = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t) = e^{-t}(C_1 \cos t + C_2 \sin t)$$

The particular solution $y_p = A_1 \cos 2t + A_2 \sin 2t$ has derivatives

$$y_p' = -2A_1 \sin 2t + 2A_2 \cos 2t, \quad y_p'' = -4A_1 \cos 2t - 4A_2 \sin 2t,$$

which when subsitituted into the equation provides

$$y_p'' + 2y_p' + 2y_p = 10\cos 2t \quad \Rightarrow \quad (-4A_1 \cos 2t - 4A_2 \sin 2t)$$
$$+ 2(-2A_1 \sin 2t + 2A_2 \cos 2t) + 2(A_1 \cos 2t + A_2 \sin 2t) = 10 \cos 2t$$
$$\Rightarrow \quad -A_1 + 2A_2 = 5, \quad 2A_1 + A_2 = 0 \quad \Rightarrow \quad A_1 = -1, \quad A_2 = 2.$$

Thus, a particular solution is $y_p = -\cos 2t + 2\sin 2t$. This leads to the general solution

$$y(t) = y_h(t) + y_p(t) = e^{-t}(C_1 \cos t + C_2 \sin t) - \cos 2t + 2\sin 2t$$

ICs: $y(0) = 0 = C_1 - 1$ and $y'(0) = 1 = C_2 - C_1 + 4$ imply

$$C_1 = 1, \quad C_2 = -2 \quad \Rightarrow y(t) = e^{-t}(\cos t - 2\sin t) - \cos 2t + 2\sin 2t$$

25 points 2. Find the solution of the Logistic equation

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P, \quad P(0) = P_0,$$

where r, K and P_0 are constants.
Solution: The logistic equation is separable. We can write

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P \quad \Longrightarrow \quad \frac{K}{P(K-P)}dP = rdt$$

We use partial fraction to discover that

$$\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}$$

Hence we solve

$$\left(\frac{1}{P} + \frac{1}{K-P}\right)dP = rdt$$

Integrating both sides, we get

$$\ln|P| - \ln|K-P| = rt + C \quad \Longrightarrow \quad \ln\left|\frac{P}{K-P}\right| = rt + C$$

or

$$\left|\frac{P}{K-P}\right| = e^{rt+C} = e^C e^{rt}$$

The constant e^C is positive. If we replace it by A, and allow it to be positive, negative, or zero, we can drop the absolute values and write

$$\frac{P}{K-P} = Ae^{rt}$$

Let $P(0) = P_0$, we have

$$A = \frac{P_0}{K - P_0}$$

Next we solve the equation for P to get

$$P(t) = \frac{KP_0}{P_0 + (K-P_0)e^{-rt}}$$

25 points 3. Suppose a population is growing according to the logistic equation

$$\frac{dx}{dt} = f(x) \quad \text{where } f(x) = x - x^2$$

Perform each of the following tasks without the aid of technology.

(i) Sketch a graph of $f(x)$

(ii) Use the graph of f to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

(iii) Sketch the equilibrium solutions in the t-x plans. These equilibrium solutions divide the t-x plane into regions. Sketch at least one solution trajectory in each of these regions.

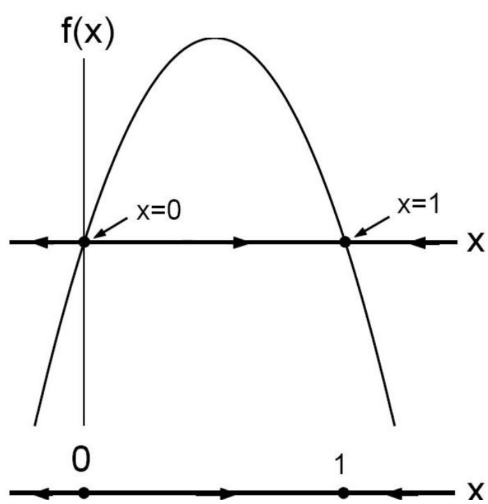
Solution (i) The graph of $f(x)$ and (ii) the associated phase line are shown in Figure (left). (ii) The equilibrium points are where $f(x) = 0$, or at $x_1 = 0$ and $x_2 = 1$. Note that

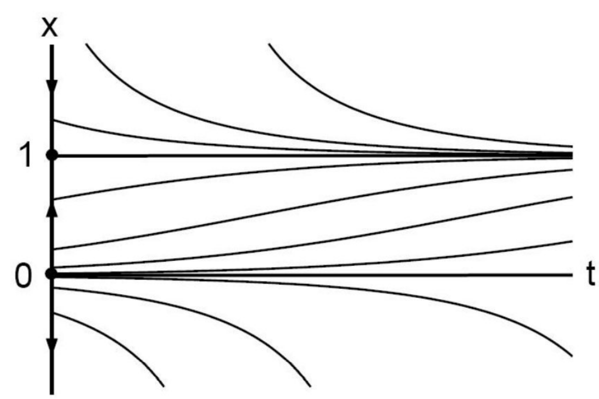
$$f(x) > 0 \text{ if } 0 < x < 1 \quad \text{and} \quad f(x) < 0 \text{ if } P < 0 \text{ or } P > 1$$

Hence $x_1 = 0$ is un unstable equilibrium point and $x_2 = 1$ is stable. (iii) The equilibrium solutions are

$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = 1.$$

The graph of the equilibrium solutions are shown in Figure (rigth). The solution curves are sketched in Figure (right), where any solution with a positive population must stay positive and must tend to 1 as $t \mapsto \infty$.





20 points

4. (**BONUS PROBLEM**) An undamped spring-mass system with external driving force is modeled with

$$x'' + 25x = 4\cos 5t.$$

The parameters of this equation are “tuned” so that the frequency of the driving force equals the natural frequency of the undriven system. Suppose that the mass is displaced one positive unit and released from rest.

(a) Find the position of the mass as a function of time. What part of the solution guarantees that this solution resonates?

(b) Sketch the solution found in part (a).

Solution (a) As in the notes, the particular solution is

$$x_p(t) = \frac{A}{2\omega_0} t \sin \omega_0 t = \frac{2}{5} t \sin 5t$$

The general solution of the homogeneous equation is

$$x_h(t) = C_1 \cos 5t + C_2 \sin 5t$$

So the solution has the form

$$x(t) = x_h(t) + x_p(t) = C_1 \cos 5t + C_2 \sin 5t + \frac{2}{5} t \sin 5t$$

Apply I.C.s yields $1 = x(0) = C_1$ and $0 = x'(0) = 5C_2$. So

$$x(t) = \cos 5t + \frac{2}{5} t \sin 5t$$

The particular solution $x_p(t)$ has a factor of t so its amplitude will grow, indicating a resonant solution.
(b)