ODE

Sample Midterm 1 Math 3331 (Summer 2014) — June 16, 2014

50 points — 1. Find the solution of the following initial-value problem

1. $\dfrac{dy}{dt} = ty^2, \quad y(0) = 1.$
2. $\dfrac{dy}{dt} = ry + a, \quad y(0) = y_0.$ (r, a, y_0 parameters)
3. $\dfrac{dy}{dt} = \dfrac{y}{t}, \quad y(1) = -2.$
4. $\dfrac{dy}{dt} = \dfrac{\sin t}{y}, \quad y(\dfrac{\pi}{2}) = 1.$
5. $\dfrac{dy}{dt} = 1 + y^2, \quad y(0) = 1.$
6. $\dfrac{dy}{dt} = \dfrac{t}{y}, \quad y(0) = 1.$
7. $\dfrac{dy}{dt} + \cos ty = \dfrac{1}{2}\sin 2t, \quad y(0) = 1.$
8. $\dfrac{dy}{dt} + 2ty = 2t^3, \quad y(0) = -1.$
9. $\dfrac{dy}{dt} + \dfrac{y}{1+t} = 2, \quad y(0) = 1.$
10. $\dfrac{dy}{dt} - \dfrac{n}{t}y = e^t t^n, \quad y(1) = 1.$

50 points — 2. Find the general solution of the following differential equations

- (Radiactive decay)
$$\frac{dN}{dt} = -\lambda N$$
where λ is a constant.
- (Newton's law of cooling)
$$\frac{dT}{dt} = k(A - T)$$
where k and A are constants.
- (Motion with air resistance)
$$\frac{dv}{dt} = -g - \frac{k}{m}v$$
where g, k and m are constants.
- (Personal finance)
$$\frac{dP}{dt} = rP + Q(t),$$
where $Q(t)$ is a given function of t.

- (Logistic equation)
$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P,$$
where k and N are constants.

20 points 3. Suppose a population is growing according to the logistic equation

$$\frac{dP}{dt} = f(P) \quad \text{where } f(P) = r_0\left(1 - \frac{P}{K}\right)P$$

with r_0 being the natural reproductive rate and K being the carrying capacity. Perform each of the following tasks without the aid of technology.

(i) Sketch a graph of $f(P)$

(ii) Use the graph of f to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

(iii) Sketch the equilibrium solutions in the t-P plans. These equilibrium solutions divide the t-P plane into regions. Sketch at least one solution trajectory in each of these regions.

20 points 4. A population is observed to obey the logistic equation with eventual population 20,000. The initial population is 1,000, and 8 hour later, the observed population is 1,200. Find the reproductive rate and the time required for the population to reach 75% of its carrying capacity.

20 points 5. On the day of his birth, Jason's grandmother pledges to make available \$50,000 on his eighteenth birthday for his college education. She negotiate an account paying 6.25% annual interest, compounded continuously, with no initial deposit, but agrees to deposit a fixed amount each year. What annual deposit should be made to reach her goal?

20 points 6. Don and Heidi would like to buy a home. They've examined their budget and determined that they can afford monthly payments of \$1,000. If the annual interest is 7.25%, and the term of the loan is 30 years, what amount can they afford to borrow?

20 points 7. Find the solution of the following initial-value problem

a. $y'' - 3y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1.$

b. $y'' + 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 3.$

c. $y'' - 2y' + y = 0, \quad y(0) = 2, \quad y'(0) = -1.$

20 points 8. A 0.1 kg mass is attached to a spring having a spring constant 3.6 kg/s. The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present, find the amplitude, frequency, and phase of the resulting motion. Plot the solution.

$\boxed{\text{20 points}}$ 9. Find the solution of the following initial-value problems

a. $y'' + 3y' + 2y = 3e^{-4t}, \quad y(0) = 1, \quad y'(0) = 0.$

b. $y'' + 2y' + 2y = 2\cos 2t, \quad y(0) = -2, \quad y'(0) = 0.$

c. $y'' - 2y' + y = t^3, \quad y(0) = 1, \quad y'(0) = 0.$

d. $y'' + 4y' + 4y = 2e^{-2t}, \quad y(0) = 0, \quad y'(0) = 1.$

$\boxed{\text{20 points}}$ 10. Verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2y'' + 3ty' - 3y = 0, \qquad \text{for } t > 0.$$

Use the variation of parameters to find the general solution to

$$t^2y'' + 3ty' - 3y = \frac{1}{t}, \qquad \text{for } t > 0.$$

$\boxed{\text{20 points}}$ 11. An undamped spring-mass system with external driving force is modeled with

$$x'' + 25x = 4\cos 5t.$$

The parameters of this equation are "tuned" so that the frequency of the driving force equals the natural frequency of the undriven system. Suppose that the mass is displaced one positive unit and released from rest.

(a) Find the position of the mass as a function of time. What part of the solution guarantees that this solution resonates?

(b) Sketch the solution found in part (a).