Name and ID: ______________________________

$\boxed{\text{50 points}}$ 1. Use the Laplace Transform to find the solution of the following initial-value problems

a. $y'' + y = \sin 2t, \quad y(0) = 0, \quad y'(0) = 1.$

Solution Let $Y(s) = \mathcal{L}(y(t))$. Then

$$\mathcal{L}(y'' + y) = \mathcal{L}\{\sin 2t\} \quad \Rightarrow \quad \mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}\{\sin 2t\}$$
$$\Rightarrow \quad \big(s^2 Y - s y(0) - y'(0)\big) + Y = \frac{2}{s^2+4}$$
$$\text{I.Cs.} \quad \Rightarrow \quad (s^2+1)Y - 1 = \frac{2}{s^2+4} \quad \Rightarrow \quad Y(s) = \frac{s^2+6}{(s^2+1)(s^2+4)}$$

Note that the partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{s^2+6}{(s^2+1)(s^2+4)} = \frac{5}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}$$

Then

$$y(t) = \mathcal{L}^{-1}(Y(s))(t) = \frac{5}{3}\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)$$
$$= \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

b. $y'' - y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$

Solution Let $Y(s) = \mathcal{L}(y(t))$. Then

$$\mathcal{L}(y'' - y) = \mathcal{L}\{e^{-t}\} \quad \Rightarrow \quad \mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}\{e^{-t}\}$$
$$\Rightarrow \quad \big(s^2 Y - s y(0) - y'(0)\big) - Y = \frac{1}{s+1}$$
$$\text{I.Cs.} \quad \Rightarrow \quad (s^2-1)Y = \frac{1}{s+1} \quad \Rightarrow \quad Y(s) = \frac{1}{(s-1)(s+1)^2}$$

Note that the partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{1}{(s-1)(s+1)^2} = \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} - \frac{1}{2}\frac{1}{(s+1)^2}$$

Then

$$y(t) = \mathcal{L}^{-1}(Y(s))(t) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)$$
$$= \frac{1}{4}e^{t} - \frac{1}{4}e^{-t} - \frac{1}{2}te^{-t}$$

50 points 2. Consider the initial value problem

$$x' = x + t, \quad x(0) = 1. \tag{1}$$

Carry out one step calculation of the Euler, RK2 and RK4 methods with step size $h = 0.5$ to approximate the value of $x(0.5)$ and compute the error of your numerical solution.

Solution We have $t_0 = 0$, $x_0 = 1$, $h = \frac{1}{2}$, and $f(t, y) = t + x$.

1. The first step of Euler's method is completed as follows

$$\begin{aligned} s_1 &= f(t_0, x_0) = f(0, 1) = 0 + 1 = 1 \\ x_1 &= x_0 + hs_1 = 1 + \frac{1}{2}1 = \frac{3}{2} = 1.5 \\ t_1 &= t_0 + h = 0 + \frac{1}{2} = \frac{1}{2} = 0.5 \end{aligned}$$

2. The first step of RK2 method follows. First we compute the slopes

$$\begin{aligned} s_1 &= f(t_0, x_0) = f(0, 1) = 0 + 1 = 1 \\ s_2 &= f(t_0 + h, x_0 + hs_1) = f(\frac{1}{2}, \frac{3}{2}) = \frac{1}{2} + \frac{3}{2} = 2 \end{aligned}$$

You can now update x and t

$$\begin{aligned} x_1 &= x_0 + h\frac{1}{2}(s_1 + s_2) = 1 + \frac{1}{2} \cdot \frac{1}{2}(1 + 2) = \frac{7}{4} = 1.75 \\ t_1 &= t_0 + h = 0 + \frac{1}{2} = \frac{1}{2} = 0.5 \end{aligned}$$

3. The first step of RK4 method follows. First we compute the four slopes

$$\begin{aligned} s_1 &= f(t_0, x_0) = f(0, 1) = 0 + 1 = 1 \\ s_2 &= f(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}s_1) = f(\frac{1}{4}, \frac{5}{4}) = \frac{1}{4} + \frac{5}{4} = \frac{3}{2}(= 1.5) \\ s_3 &= f(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}s_2) = f(\frac{1}{4}, \frac{11}{8}) = \frac{1}{4} + \frac{11}{8} = \frac{13}{8}(= 1.625) \\ s_4 &= f(t_0 + h, x_0 + hs_3) = f(\frac{1}{2}, \frac{29}{16}) = \frac{1}{2} + \frac{29}{16} = \frac{37}{16}(= 2.3125) \end{aligned}$$

You can now update x and t

$$\begin{aligned} x_1 &= x_0 + h\frac{1}{6}(s_1 + 2(s_2 + s_3) + s_4) \\ &= 1 + \frac{1}{2} \cdot \frac{1}{6}\left(1 + 2\left(\frac{3}{2} + \frac{13}{8}\right) + \frac{37}{16}\right) = 1 + \frac{51}{64} \approx 1.7969 \\ t_1 &= t_0 + h = 0 + \frac{1}{2} = \frac{1}{2} = 0.5 \end{aligned}$$

4. The equation is linear and we can find its solution $x(t) = 2e^t - t - 1$ (note that $x(t) = x_h(t) + x_p(t)$ with $x_h(t) = ce^t$ and $x_p(t) = -t - 1$). We can compute the true value: $x(0.5) = 2e^{0.5} - 0.5 - 1 \approx 1.7974$.

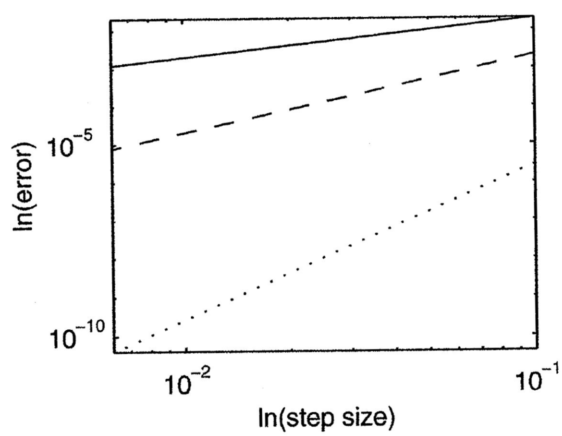
5. We can complete the following table

	time	approx.	true value	error
Euler	0.5	1.5	1.7974	0.2974
RK2	0.5	1.75	1.7974	0.0474
RK4	0.5	1.7969	1.7974	0.0005

20 points 3. Consider the initial value problem

$$x' = -tx^2, \quad 0 \le t \le 2, \quad x(0) = 3. \tag{2}$$

The equation is separable and the solution is $x(t) = 6/(3t^2+2)$. We used the Euler, RK2 and RK4 methods to compute the value of $x(2)$ and constructed a plot of the logarithm of the error versus the logarithm of the step size for each numerical method. The slope of the solid line is 1.0135, the slope of the dashed line is 2.0303, and the slope of the dotted line is 4.0256. Indicate each line by its corresponding numerical method and explain your answer.



Solution

1. Euler's method produces the results shown in the solid line, as the slope of the solid line is 1.0135, which is closed to 1, consistent with the fact that the Euler method is a first order algorithm.

2. RK2 method produces the results shown in the dashed line as the slope of the dashed line is 2.0303, which is close to 2, consistent with the fact that RK2 is a second order method.

3. Finally, RK4 method produces the results shown in the dotted line as the slope of the dotted line is 4.0256, which is close to 4, consistent with the fact that RK4 is a fourth order method.