ODE

Sample Midterm 3 Math 3331 (Summer 2014) **July 2, 2014**

30 points 1. Find the solution of the initial-value problem

$$
\begin{aligned}
x' &= -3x - z \\
y' &= 3x + 2y + 3z \\
z' &= 2x
\end{aligned}
$$

with $x(0) = 1$, $y(0) = -1$ and $z(0) = 2$.

Solution In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -3 & 0 & -1 \\ 3 & 2 & 3 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigen-pairs of $A = \begin{pmatrix} -3 & 0 & -1 \\ 3 & 2 & 3 \\ 2 & 0 & 0 \end{pmatrix}$ are

$$
\begin{aligned}
&\lambda_1 = -2, \quad \lambda_2 = -1, \quad \lambda_3 = 2 \\
&v_1 = (-1, 0, 1)^T, \quad v_2 = (1, 1, -2)^T, \quad v_3 = (0, 1, 0)^T.
\end{aligned}
$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

If $x(0) = 1$, $y(0) = -1$ and $z(0) = 2$, then

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

We find that $c_1 = -4$, $c_2 = -3$, and $c_3 = 2$. Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = -4e^{-2t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - 3e^{-t} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + 2e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4e^{-2t} - 3e^{-t} \\ 3e^{-t} + 2e^{2t} \\ -4e^{-2t} + 6e^{-t} \end{pmatrix}$$

30 points 2. Find the solution of the initial-value problem

$$
\begin{aligned}
x' &= -3x \\
y' &= -5x + 6y - 4z \\
z' &= -5x + 2y
\end{aligned}
$$

with $x(0) = -2$, $y(0) = 0$ and $z(0) = 2$.

Solution In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -3 & 0 & 0 \\ -5 & 6 & -4 \\ -5 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigen-pairs of $A = \begin{pmatrix} -3 & 0 & 0 \\ -5 & 6 & -4 \\ -5 & 2 & 0 \end{pmatrix}$ are

$$\lambda_1 = 4, \quad \lambda_2 = -3, \quad \lambda_3 = 2$$
$$v_1 = (0, 2, 1)^T, \quad v_2 = (1, 1, 1)^T, \quad v_3 = (0, 1, 1)^T.$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{4t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

If $x(0) = -2$, $y(0) = 0$ and $z(0) = 2$, then

$$\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

We find that $c_1 = -2$, $c_2 = -2$, and $c_3 = 6$. Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = -2e^{4t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - 2e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 6e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2e^{-3t} \\ -4e^{4t} - 2e^{-3t} + 6e^{2t} \\ -2e^{4t} - 2e^{-3t} + 6e^{2t} \end{pmatrix}$$

30 points

3. Find the solution of the initial-value problem

$$x' = -4x + 8y + 8z$$
$$y' = -4x + 4y + 2z$$
$$z' = 2z$$

with $x(0) = 1$, $y(0) = 0$ and $z(0) = 0$.

Solution In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -4 & 8 & 8 \\ -4 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigen-pairs of $A = \begin{pmatrix} -4 & 8 & 8 \\ -4 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ are

$$\begin{aligned} &\lambda_1 = 2, \quad \lambda_2 = 4i, \quad \lambda_3 = -4i \\ &v_1 = (0,-1,1)^T, \quad v_2 = (1-i,1,0)^T, \quad v_3 = (1+i,1,0)^T. \end{aligned}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \cos 4t + \sin 4t \\ \cos 4t \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -\cos 4t + \sin 4t \\ \sin 4t \\ 0 \end{pmatrix}$$

If $x(0) = 1$, $y(0) = 0$ and $z(0) = 0$, then

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

We find that $c_1 = 0$, $c_2 = 0$, and $c_3 = -1$. Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos 4t - \sin 4t \\ -\sin 4t \\ 0 \end{pmatrix}$$

30 points

4. Find the solution of the initial-value problem

$$\begin{aligned} x' &= 6x - 4z \\ y' &= 8x - 2y \\ z' &= 8x - 2z \end{aligned}$$

with $x(0) = -2$, $y(0) = -1$ and $z(0) = 0$.

Solution In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 6 & 0 & -4 \\ 8 & -2 & 0 \\ 8 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigen-pairs of $A = \begin{pmatrix} 6 & 0 & -4 \\ 8 & -2 & 0 \\ 8 & 0 & -2 \end{pmatrix}$ are

$$\begin{aligned} &\lambda_1 = -2, \quad \lambda_2 = 2+4i, \quad \lambda_3 = 2-4i \\ &v_1 = (0,1,0)^T, \quad v_2 = (1+i,2,2)^T, \quad v_3 = (1-i,2,2)^T. \end{aligned}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \cos 4t - \sin 4t \\ 2\cos 4t \\ 2\cos 4t \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} \cos 4t + \sin 4t \\ 2\sin 4t \\ 2\sin 4t \end{pmatrix}$$

If $x(0) = -2$, $y(0) = -1$ and $z(0) = 0$, then

$$\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We find that $c_1 = -1$, $c_2 = 0$, and $c_3 = -2$. Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = -e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2e^{2t} \begin{pmatrix} \cos 4t + \sin 4t \\ 2\sin 4t \\ 2\sin 4t \end{pmatrix} = \begin{pmatrix} -2e^{2t}(\cos 4t + \sin 4t) \\ -e^{-2t} - 4e^{2t}\sin 4t \\ -4e^{2t}\sin 4t \end{pmatrix}$$

30 points

5. Find the general solution of the system

$$\begin{aligned} x' &= 6x - 5y + 10z \\ y' &= -x + 2y - 2z \\ z' &= -x + y - z \end{aligned}$$

Solution In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 6 & -5 & 10 \\ -1 & 2 & -2 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigen-pairs of $A = \begin{pmatrix} 6 & -5 & 10 \\ -1 & 2 & -2 \\ -1 & 1 & -1 \end{pmatrix}$ are

$$\lambda_1 = 5, \quad \lambda_2 = \lambda_3 = 1 \qquad v_1 = (-5, 1, 1)^T, \quad v_2 = (1, 1, 0)^T, \quad v_3 = (-2, 0, 1)^T.$$

Note that the eigenvalue $\lambda_2 = 1$ has geometric multiplicity 2 equal to its algebraic multiplicity. The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{5t} \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{t} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

30 points 6. Find the general solution of the system

$$x' = -2x + y - z$$
$$y' = x - 3y$$
$$z' = 3x - 5y$$

Solution In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & 0 \\ 3 & -5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigenvalues of $A = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & 0 \\ 3 & -5 & 0 \end{pmatrix}$ are

$$\lambda_1 = -1, \quad \lambda_2 = \lambda_3 = -2$$

The eigenvalue $\lambda_1 = -1$ has geometric multiplicity 1, and an eigenvector is $v_1 = (-2, -1, 1)^T$, leading to the solution

$$y_1(t) = e^{tA} v_1 = e^{\lambda_1 t} v_1 = e^{-t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

The eigenvalue $\lambda_2 = -2$ has geometric multiplicity 1, less than its algebraic multiplicity, and an eigenvector is $v_2 = (1, 1, 1)^T$, leading to the solution

$$y_2(t) = e^{tA} v_2 = e^{\lambda_2 t} v_2 = e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Note that

$$(A - \lambda_2 I)^2 = (A + 2I)^2 = \begin{pmatrix} -2 & 4 & -2 \\ -1 & 2 & -1 \\ 1 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus null$((A - \lambda_2 I)^2)$ has dimension two, equalling its algebraic multiplicity. We pick a vector in the nullspace of $(A - \lambda_2 I)^2)$ that is not in the nullspace of $A - \lambda_2$, for example, $v_3 = (-1, 0, 1)^T$, leading to the solution

$$\begin{aligned} y_3(t) &= e^{tA} v_3 = e^{\lambda_2 t}(v_3 + t(A - \lambda_2 I) v_3) \\ &= e^{-2t} \left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= e^{-2t} \begin{pmatrix} -1 - t \\ -t \\ 1 - t \end{pmatrix} \end{aligned}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = c_1 e^{-t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} -1-t \\ -t \\ 1-t \end{pmatrix}$$

30 points 7. Classify the equilibrium point of the system $y' = Ay$. Sketch the phase portrait by hand.

$$(1)\ A = \begin{pmatrix} -16 & 9 \\ -18 & 11 \end{pmatrix} \qquad (2)\ A = \begin{pmatrix} 8 & 3 \\ -6 & -1 \end{pmatrix} \qquad (3)\ A = \begin{pmatrix} -11 & -5 \\ 10 & 4 \end{pmatrix}$$

$$(4)\ A = \begin{pmatrix} 2 & -4 \\ 8 & 6 \end{pmatrix} \qquad (5)\ A = \begin{pmatrix} 6 & -5 \\ 10 & -4 \end{pmatrix} \qquad (6)\ A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$$

$$(7)\ A = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix} \qquad (8)\ A = \begin{pmatrix} -4 & -4 \\ 1 & 0 \end{pmatrix} \qquad (9)\ A = \begin{pmatrix} 2 & 1 \\ -10 & -5 \end{pmatrix}$$

Solution (1) If

$$A = \begin{pmatrix} -16 & 9 \\ -18 & 11 \end{pmatrix}$$

then the trace is $T = -5$ and the deternimant is $D = -14 < 0$. Hence, the equilibrium point at the origin is a saddle. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 5\lambda - 14$$

which produces eigenvalues $\lambda_1 = -7$ and $\lambda_2 = 2$. Because

$$A - \lambda_1 I = A + 7I = \begin{pmatrix} -9 & 9 \\ -18 & 18 \end{pmatrix} \to v_1 = (1,1)^T$$

leading to the exponential solution

$$y_1(t) = e^{\lambda_1 t} v_1 = e^{-7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Because

$$A - \lambda_2 I = A - 2I = \begin{pmatrix} -18 & 9 \\ -18 & 9 \end{pmatrix} \to v_2 = (1,2)^T$$

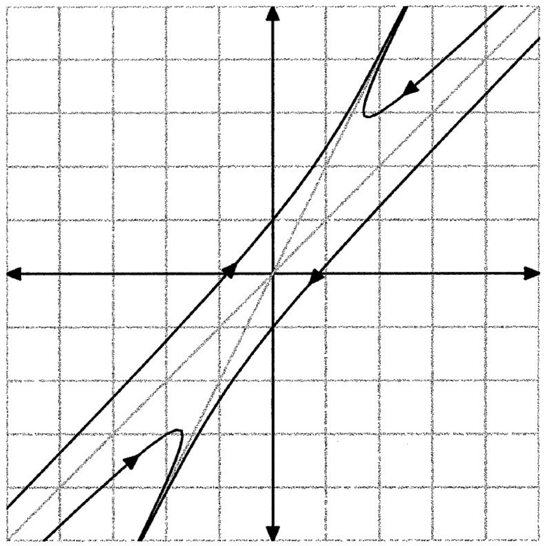
leading to the exponential solution

$$y_2(t) = e^{\lambda_2 t} v_2 = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solutions approach the halfline generated by $c_2(1,2)^T$ as they move forward in time, but they approach the halfline generated by $c_1(1,1)^T$ as they move backward in time. A hand sketch follows



(2) If

$$A = \begin{pmatrix} 8 & 3 \\ -6 & -1 \end{pmatrix}$$

then the trace is $T = 7$ and the deternimant is $D = 10 > 0$. Further, $T^2 - 4D = 9 > 0$, so the equilibrium point at the origin is a nodal source. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 7\lambda + 10$$

which produces eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. Because

$$A - \lambda_1 I = A - 2I = \begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix} \to v_1 = (1, -2)^T$$

leading to the exponential solution

$$y_1(t) = e^{\lambda_1 t} v_1 = e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Because

$$A - \lambda_2 I = A - 5I = \begin{pmatrix} 3 & 3 \\ -6 & -6 \end{pmatrix} \to v_2 = (1, -1)^T$$

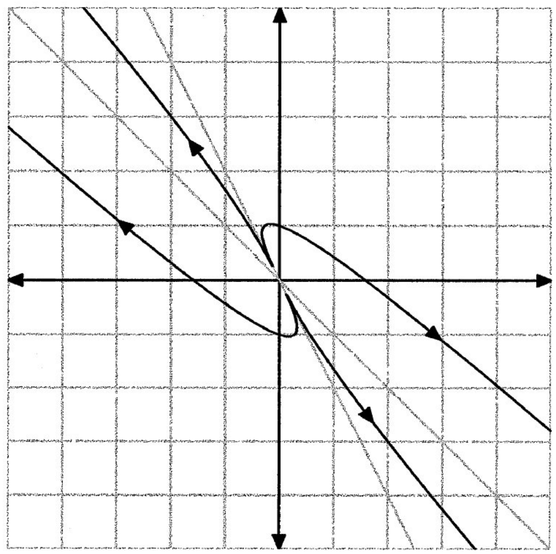
leading to the exponential solution

$$y_2(t) = e^{\lambda_2 t} v_2 = e^{5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solutions emanate from the source tangent to the "slow" halfline generated by $c_1(1, -2)^T$ and eventually parallel the "fast" halfline generated by $c_2(1, -1)^T$ as they move forward in time. A hand sketch follows



(3) If

$$A = \begin{pmatrix} -11 & -5 \\ 10 & 4 \end{pmatrix}$$

then the trace is $T = -7$ and the deternimant is $D = 6 > 0$. Further, $T^2 - 4D = 25 > 0$, so the equilibrium point at the origin is a nodal sink. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 7\lambda + 6$$

which produces eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -6$. Because

$$A - \lambda_1 I = A + I = \begin{pmatrix} -10 & -5 \\ 10 & 5 \end{pmatrix} \to v_1 = (1, -2)^T$$

leading to the exponential solution

$$y_1(t) = e^{\lambda_1 t} v_1 = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Because

$$A - \lambda_2 I = A + 6I = \begin{pmatrix} -5 & -5 \\ 10 & 10 \end{pmatrix} \to v_2 = (1, -1)^T$$

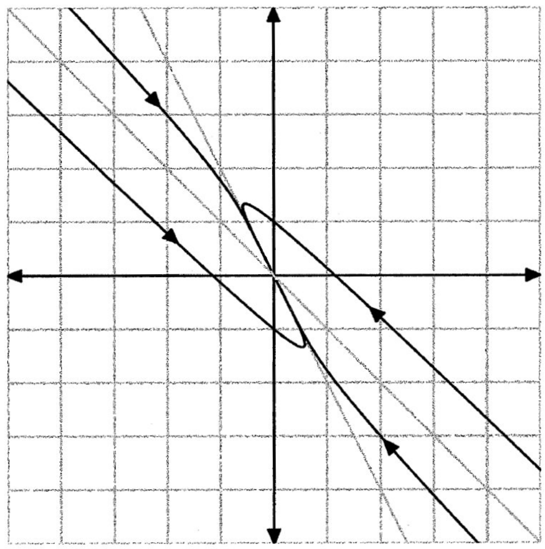
leading to the exponential solution

$$y_2(t) = e^{\lambda_2 t} v_2 = e^{-6t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solutions approach the origin tangent to the "slow" halfline generated by $c_1(1,-2)^T$. As time move backward, solutions eventually parallel the "fast" halfline generated by $c_2(1,-1)^T$. A hand sketch follows



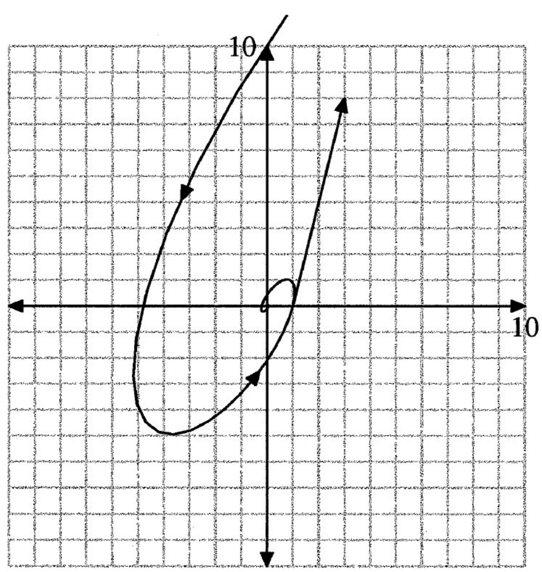
(4) If

$$A = \begin{pmatrix} 2 & -4 \\ 8 & 6 \end{pmatrix}$$

then the trace is $T = -4$ and the deternimant is $D = 20 > 0$. Further, $T^2 - 4D = -64 < 0$, so the equilibrium point at the origin is a spiral sink. At $(1,0)$,

$$\begin{pmatrix} 2 & -4 \\ 8 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

so the motion is counterclockwise. A hand sketch follows



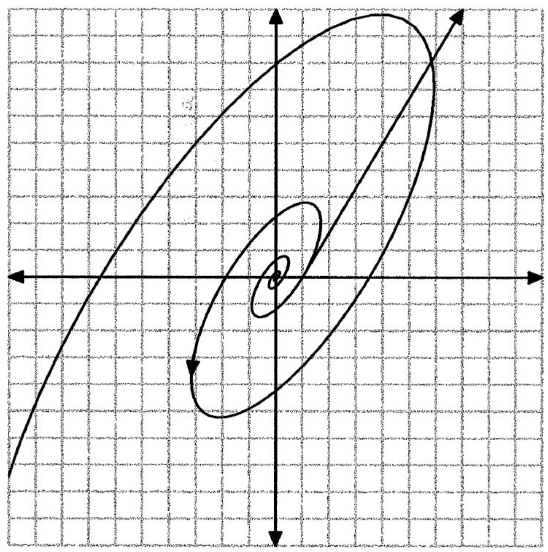
(5) If

$$A = \begin{pmatrix} 6 & -5 \\ 10 & -4 \end{pmatrix}$$

then the trace is $T = 2$ and the deternimant is $D = 26 > 0$. Further, $T^2 - 4D = -100 < 0$, so the equilibrium point at the origin is a spiral source. At $(1, 0)$,

$$\begin{pmatrix} 6 & -5 \\ 10 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$$

so the motion is counterclockwise. A hand sketch follows



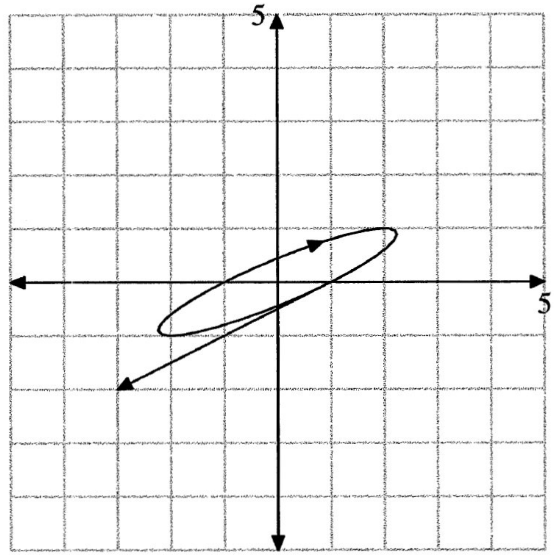
(6) If

$$A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$$

then the trace is $T = 0$ and the deternimant is $D = 4$, so the equilibrium point at the origin is a center. At $(1, 0)$,

$$\begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

so the rotation is clockwise. A hand sketch follows



(7) If

$$A = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$$

then the trace is $T = 8 > 0$ and the deternimant is $D = 16 > 0$. Further, $T^2 - 4D = 0$, so the equilibrium point at the origin is a degenerate nodal source. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - 8\lambda + 16$$

which produces a single eigenvalue $\lambda = 4$. Because

$$A - \lambda I = A - 4I = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \to v_1 = (2, -1)^T$$

leading to the exponential solution

$$y_1(t) = e^{\lambda t} v_1 = e^{4t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

To find another solution, we must solve $(A - \lambda I)v_2 = v_1$. Start with any vector that is not a multiple of v_1, say $w = (1, 0)^T$. Then

$$(A - \lambda I)w = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = v_1$$

Thus, let $v_2 = w = (1, 0)^T$. A second, independent solution is

$$y_2(t) = e^{\lambda t}(v_2 + tv_1) = e^{4t} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right)$$

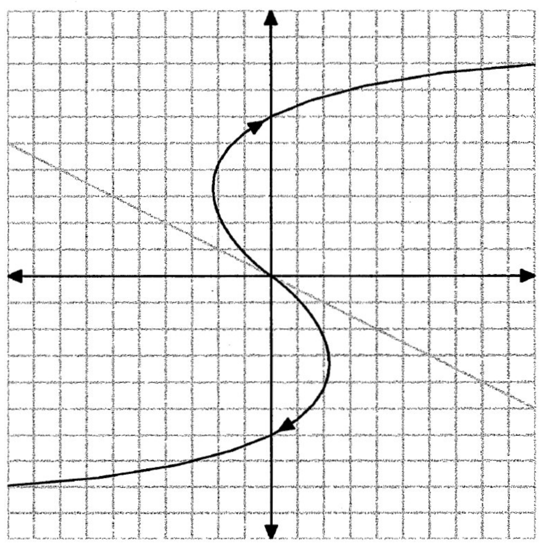
The general solution is

$$\begin{aligned} y(t) = c_1 y_1(t) + c_2 y_2(t) &= c_1 e^{4t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{4t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right) \\ &= e^{4t}\left((c_1 + c_2 t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \end{aligned}$$

Solutions emanate from the origin tangent to the halfline generated by $c_1(2,-1)^T$ and eventually parallel the halfline as they move forward in time. At $(1,0)$,

$$\begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

so the rotation is clockwise. A hand sketch follows



(8) If

$$A = \begin{pmatrix} -4 & -4 \\ 1 & 0 \end{pmatrix}$$

then the trace is $T = -4 < 0$ and the deternimant is $D = 4$. Further, $T^2 - 4D = 0$, so the equilibrium point at the origin is a degenerate nodal sink. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 4\lambda + 4$$

which produces a single eigenvalue $\lambda = -2$. Because

$$A - \lambda I = A + 2I = \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix} \to v_1 = (2,-1)^T$$

leading to the exponential solution

$$y_1(t) = e^{\lambda t} v_1 = e^{-2t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

To find another solution, we must solve $(A - \lambda I)v_2 = v_1$. Start with any vector that is not a multiple of v_1, say $w = (1, 0)^T$. Then

$$(A - \lambda I)w = \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -v_1$$

Thus, let $v_2 = -w = (-1, 0)^T$. A second, independent solution is

$$y_2(t) = e^{\lambda t}(v_2 + tv_1) = e^{-2t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$$

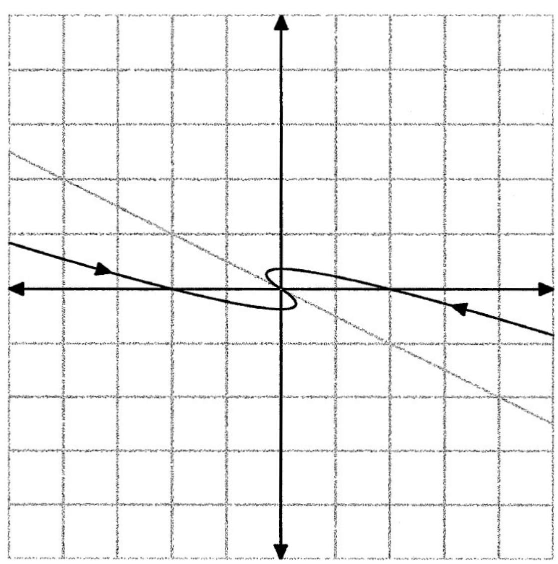
The general solution is

$$\begin{aligned} y(t) = c_1y_1(t) + c_2y_2(t) &= c_1e^{-2t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2e^{-2t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right) \\ &= e^{-2t}\left((c_1 + c_2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) \end{aligned}$$

Solutions decay to the origin tangent to the halfline generated by $c_1(2, -1)^T$. As time marches backward, the solutions also turn parallel to the halfline. At $(1, 0)$,

$$\begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

so the rotation is counterclockwise. A hand sketch follows



(9) If

$$A = \begin{pmatrix} 2 & 1 \\ -10 & -5 \end{pmatrix}$$

then the trace is $T = -3 < 0$ and the deternimant is $D = 0$. Thus, this degenerate case lies on the horizontal axis in the trace-determinant plane, separating the saddles from the nodal sinks. Further, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 3\lambda$$

which produces eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -3$. Because

$$A - \lambda_1 I = A = \begin{pmatrix} 2 & 1 \\ -10 & -5 \end{pmatrix} \to v_1 = (1, -2)^T$$

leading to the exponential solution

$$y_1(t) = e^{\lambda_1 t} v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

which produces a whole line of equilibrium points. Everything on the line generated by v_1 is an equilibrijm point. Because

$$A - \lambda_2 I = A + 3I = \begin{pmatrix} 5 & 1 \\ -10 & -2 \end{pmatrix} \to v_2 = (1, -5)^T$$

leading to the exponential solution

$$y_2(t) = e^{\lambda_2 t} v_2 = e^{-3t} \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

The general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

Each Solution in this family is the sum of a fixed multiple of $(1, -2)^T$ and a decaying multiple of $(1, -5)^T$. Thus, as $t \to \infty$, solutions move in lines parallel to $(1, -5)^T$, decaying into the line of equilibrium. A hand sketch follows