

Math 3331 Differential Equations

2.1 Differential Equations and Solutions

Jiwen He

Department of Mathematics, University of Houston

`jiwenhe@math.uh.edu`
`math.uh.edu/~jiwenhe/math3331`



2.1 ODE and Solutions

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 - Normal Form of ODE
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Formal Definition of ODE

Definition of ODE

ODE is an equation involving an unknown function y of a single variable t together with one or more of its derivatives y' , y'' , etc.

First Order ODE: General (Implicit) Form

First order ODEs often arise naturally in the form

$$\phi(t, y, y') = 0,$$

Example

$$t + 4yy' = 0.$$

This form is too general to deal with, and we will find it necessary to solve equation for y' to place it into “normal form”

$$y' = -\frac{t}{4y}$$



Normal Form of ODE

Normal Form

A first-order ODE of the form

$$y' = f(t, y)$$

is said to be in **normal form**.

Examples

$$y' = y - t$$

$$y' = -2ty$$

$$y' = y^2$$

$$y' = \cos(ty)$$



Example

Example

Place the first order ODE

$$y'^2 + y^2 = 1$$

into normal form.



Example

Example

Place the first order ODE

$$e^{y'} + y y' = 0$$

into normal form.



Solutions of ODE

Solutions of ODE

A solution of the first-order ODE

$$y' = f(t, y)$$

is a differentiable function $y(t)$ such that

$$y'(t) = f(t, y(t))$$

for all t in the interval where $y(t)$ is defined.



Check Solutions: Example

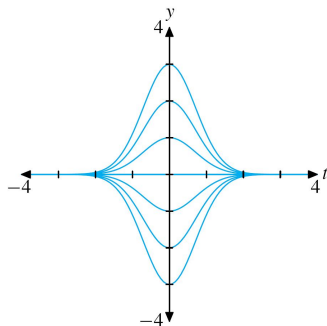
Example

Show that $y(t) = t + 1 + Ce^t$ is a solution of

$$y' = y - t.$$



General Solution and Solution Curves



Example

Show that $y(t) = Ce^{-t^2}$ is a solution of

$$y' = -2ty$$

General Solution

The solution formula $y(t) = Ce^{-t^2}$, which depends on the arbitrary constant C , describes a family of solutions and is called a **general solution**.

Solution Curves

The graphs of these solutions, drawn in the figure, are called **solution curves**.

Particular Solution

Example

- ① Show that $y(t) = 1/(C - t)$ is a general solution of

$$y' = y^2$$

- ② Find a particular solution satisfying $y(0) = 1$.

Given the value of the solution at a point, we can determine the **unique** particular solution.



Initial Value Problem

Initial Value Problem

A first-order ODE together with an initial condition,

$$y' = f(t, y), \quad y(t_0) = y_0$$

is called an **initial value problem**.

Solution of IVP

A solution of the IVP is a differentiable function $y(t)$ such that

- 1 $y'(t) = f(t, y(t))$ for all t in an interval containing t_0 where $y(t)$ is defined, and
- 2 $y(t_0) = y_0$

Example

The function $y(t) = 1/(1 - t)$ is the solution of the IVP

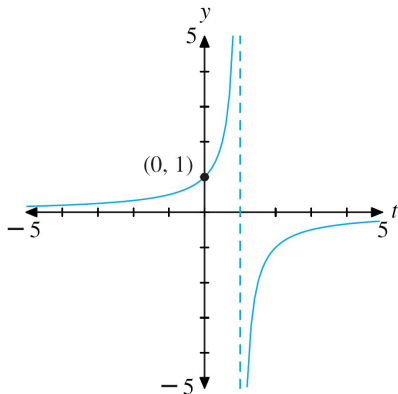
$$y' = y^2, \quad \text{with } y(0) = 1.$$



Interval of Existence

Interval of Existence

The **interval of existence** of a solution to an IVP is defined to be the largest interval over which the solution can be defined and remain a solution.



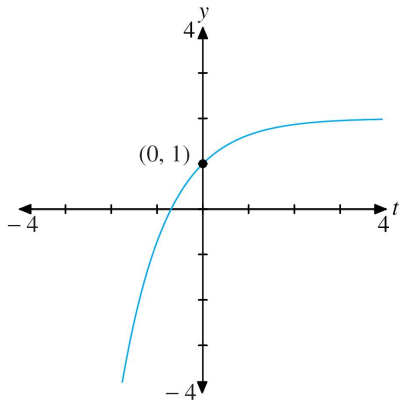
Example

Find the interval of existence for the solution to the IVP

$$y' = y^2 \quad \text{with } y(0) = 1.$$



Example



Example

- Show that $y(t) = 2 - Ce^{-t}$ is a solution of

$$y' = 2 - y$$

for any constant C .

- Find the solution that satisfies the initial condition $y(0) = 1$.
- What is the interval of existence of this solution?



Geometric Meaning of ODE

Geometric Meaning of ODE: Solution Curve and Slopes

Let $y(t)$ be a solution of the ODE

$$y' = f(t, y).$$

The graph of the solution $y(t)$ is called a **solution curve**. For any point (t_0, y_0) on the solution curve, $y(t_0) = y_0$ and the differential equation says that

$$y'(t_0) = f(t_0, y(t_0));$$

the LHS is the **slope** of the solution curve, and the RHS tells you what the slope is at (t_0, y_0) .



Direction Field

Direction Field for $y' = f(t, y)$

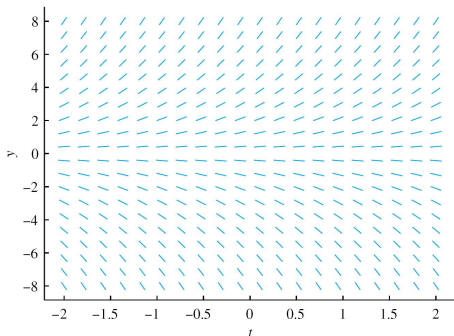
Draw a line segment with slope $f(t_i, y_j)$ attached to every grid point (t_i, y_j) in a rectangle R where $f(t, y)$ is defined

$$R = \{ (t, y) \mid a \leq t \leq b \text{ and } c \leq y \leq d \}.$$

The result is called a **direction field**.

MATLAB: `dfield6`
generated the direction
field for equation

$$y' = y$$



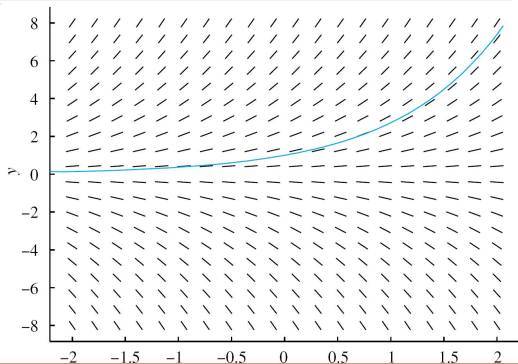
Geometric interpretation of Solutions

Direction field provides information about qualitative form of solution curves.

Finding a solution to the differential equation is equivalent to the geometric problem of finding a curve in ty -plane that is tangent to the direction field at every point.

MATLAB generated the solution curve of

$$y' = y, \quad y(0) = 1$$



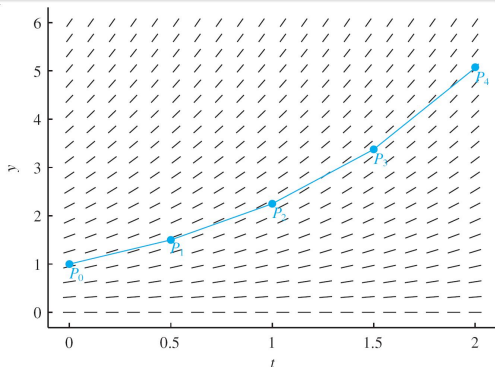
Numerical Solution of IVP: Euler's Method

Euler's Method of the Solution of IVP $y' = f(t, y)$, $y(t_0) = y_0$

- 1) Plot the point $P_0(t_0, y_0)$.
- 2) Move a prescribed distance along a line with slope $f(t_0, y_0)$ to the point $P_1(t_1, y_1)$.
- 3) Continue in this manner to produce an approximate solution curve of the IVP.

MATLAB generated an approximate solution curve of

$$y' = y, \quad y(0) = 1$$

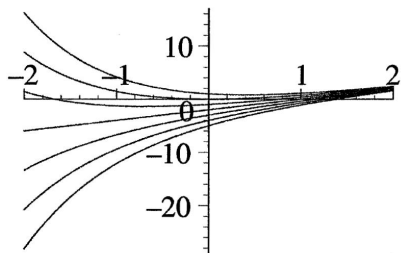


Exercise 2.4

Example (Exercise 2.4)

1). Show that the given solution is a general solution of the differential equation

$$y' + y = 2t, \quad y(t) = 2t - 2 + Ce^{-t}, \quad C = -3, -2, \dots, 3$$



- 2) Use a computer or calculator to sketch members of the family of solutions for the given values of the arbitrary constant.
- 3) Experiment with different intervals for t until you have a plot that shows what you consider to be the most important behavior of the family.

Exercise 2.13

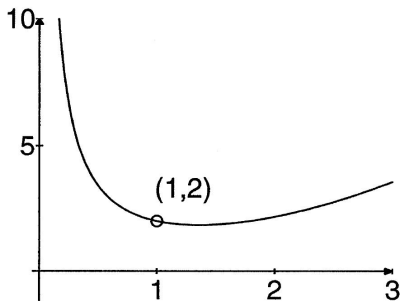
Example (Exercise 2.13)

Use the given general solution to find a solution of the differential equation having the given initial condition. Sketch the solution, the initial condition, and discuss the solutions interval of existence.

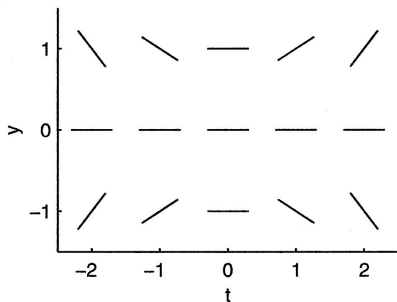
$$ty' + y = t^2, \quad y(t) = (1/3)t^2 + C/t, \quad y(1) = 2$$

$$y(t) = (1/3)t^2 + 5/(3t).$$

The interval of existence is $(0, \infty)$.



Exercise 2.19



Example (Exercise 2.19)

Plot the direction field for the differential equation by hand

$$y' = t \tan(y/2).$$

Do this by drawing short lines of the appropriate slope centered at each of the integer valued coordinates (t, y) , where $-2 \leq t \leq 2$ and $-1 \leq y \leq 1$

