Math 3331 Differential Equations

2.8 Dependence of Solutions on Initial Conditions

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- Continuity with respect to Initial Conditions
- Sensitivity to Initial Conditions

Dependence of Solutions on Initial Conditions

Q1. Continuity of the solution with respect to initial data: Can we ensure that the solution with incorrect initial data is close enough to the real solution that we can use it to predict behavior?

Q2. Sensitivity to initial conditions: Given that we have an error in the initial conditions, just how far from the true solution can the solution be?

Theorem 7.15

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Suppose the function $f(t,x)$ and its partial derivative $\frac{\partial f}{\partial x}$ are both continuous on the rectange R in the tx-plane and let

$$M = \max_{(t,x)\in R} \left|\frac{\partial f}{\partial x}\right|$$

Suppose (t_0, x_0) and (t_0, y_0) are in R and that

$$\begin{aligned} x'(t) &= f(t, x(t)), \quad \text{and } x(t_0) = x_0 \\ y'(t) &= f(t, y(t)), \quad \text{and } y(t_0) = y_0 \end{aligned}$$

Then as long as $(t, x(t))$ and $(t, y(t))$ belong to R, we have

$$|x(t) - y(t)| \leq |x_0 - y_0| e^{M|t-t_0|}$$

Example 2.8.1: Continuity w.r.t. Initial Conditions

Example 2.8.1: Consider $x' = (x-1)\cos t$. Since

$$M = \max_{(t,x)\in R}\left|\frac{\partial f}{\partial x}\right| = \max_{(t,x)\in R}|\cos t| \le 1$$

then

$$|x(t) - y(t)| \le |x_0 - y_0|e^{|t-t_0|}, \qquad \text{for all } t.$$

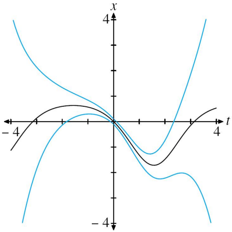

Figure 1 A solution to (8.2) with $|x(0)| \le 0.1$ must lie between the colored curves.

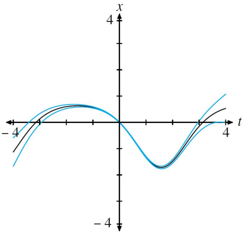

Figure 2 A solution to (8.2) with $|x(0)| \le 0.01$ must lie between the colored curves.

Example 2.8.6: Sensitivity to Initial Conditions

Example 2.8.6: Consider $x' = x \sin x + t$.

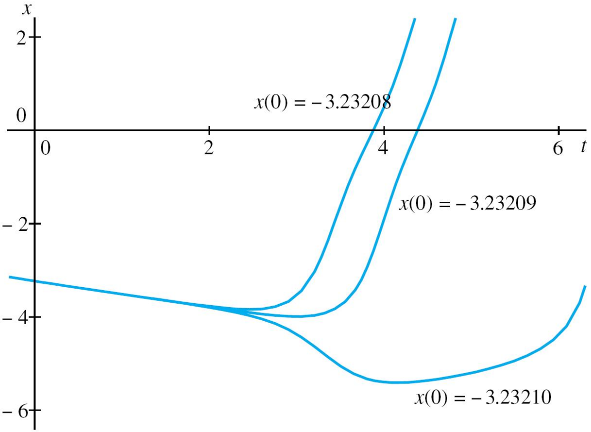

Figure 3 Sensitivity to initial conditions for solutions to $x' = x\sin(x) + t$.