Math 3331 Differential Equations

2.9 Autonomous Equations and Stability

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2.9 Autonomous Equations and Stability

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- Direction Field and Stability of Equilibrium
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- Qualitative Analysis
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Autonomous Equations

Form: $x' = f(x)$

Implicit Solution:

$$\int [1/f(x)]\, dx = \int dt$$

$$\Rightarrow \; G(x) = t + C$$

where $G(x) = \int [1/f(x)]\, dx$ is an antiderivative of $1/f(x)$

Consequence: If $x(t)$ is solution $\Rightarrow x(t + C)$ is solution

Examples

Ex: $x' = \sin(x), \;\; y' = y^2 + 1$

are autonomous

$x' = \sin(tx), \;\; y' = xy$

are *not* autonomous

Equilibrium Points and Solutions

Equilibrium Point $\mathbf{x_0}$:

Solution of $f(x_0) = 0 \Rightarrow$
$x(t) = x_0$ is constant solution

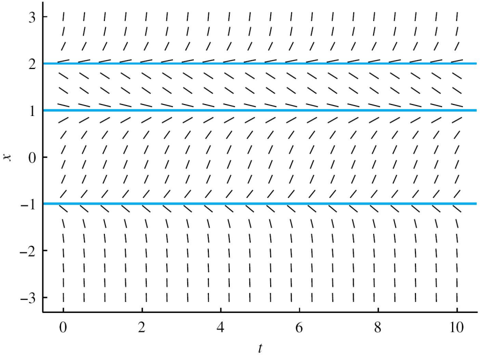
Ex.: $v' = -g - kv/m$

$f(v) = 0 \Rightarrow v_{term} = -gm/k$

is equilibrium point

(Falling Object, Air Resistance and Terminal Velocity)

Example 2.9.6



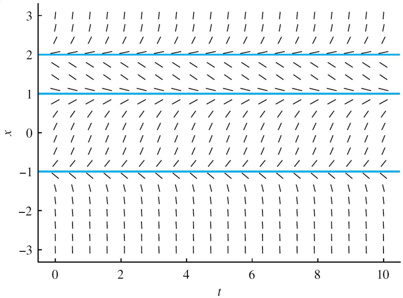
Ex.: $x' = (x^2 - 1)(x - 2)$

$$f(x) = (x-1)(x+1)(x-2) = 0$$

$$\Rightarrow \quad x_1 = 1, \quad x_2 = -1, \quad x_3 = 2$$

are equilibrium points

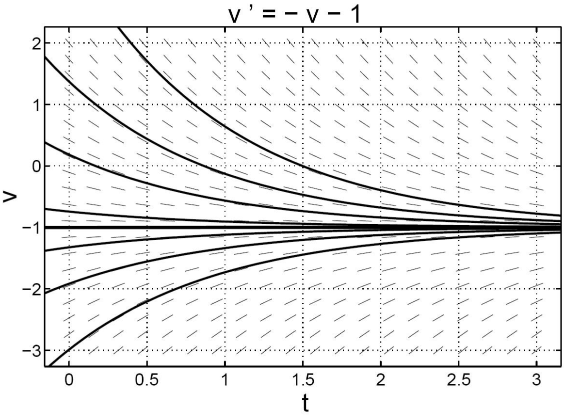
Direction Field and Stability of Equilibrium



- **Direction Field:** same slopes on horizontal lines
- **Equilibrium Solutions $\mathbf{x(t) = x_0}$:**

  $f(x_0) = 0 \Rightarrow$ solution curves are horizontal line
- **Stability of Equilibrium:** Equilibrium point x_0 is
  - *asymptotically stable* if $x(t) \to x_0$ for $t \to \infty$ when $|x(0) - x_0|$ is sufficiently small
  - *unstable* if there are solutions $x(t)$ with $|x(0)-x_0|$ arbitrarily small that move away from x_0 when t increases

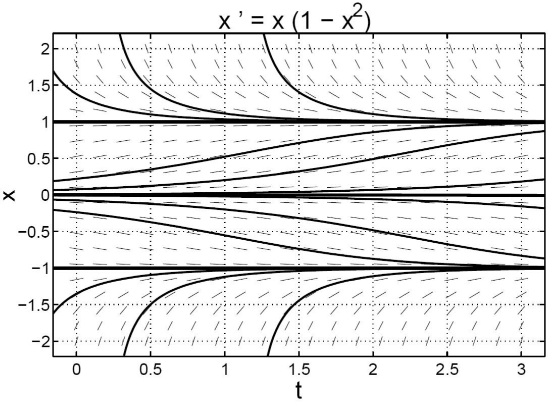
Example: Falling Object and Terminal Velocity



Ex.: $v' = -v - 1$

$v_0 = -1$: asymptotically stable

Example

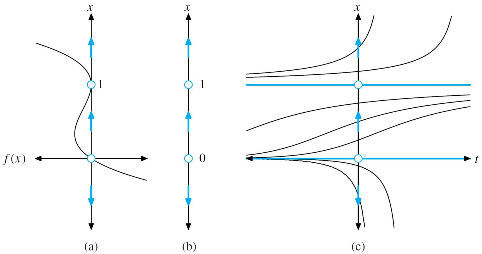


Ex.: $x' = x(1 - x^2)$

$x_1 = 0$: unstable

$x_{2,3} = \pm 1$: asymptotically stable

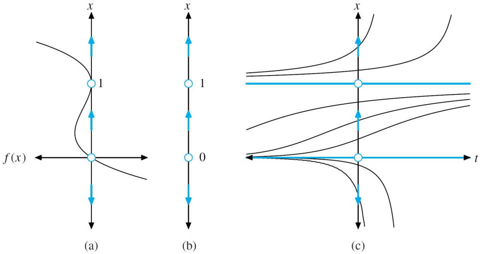
Properties of Solutions



Properties of Solutions

- Equilibrium solutions divide tx–plane into horizontal funnels
- In each funnel solutions are
  -increasing if $x' = f(x) > 0$
  -decreasing if $x' = f(x) < 0$

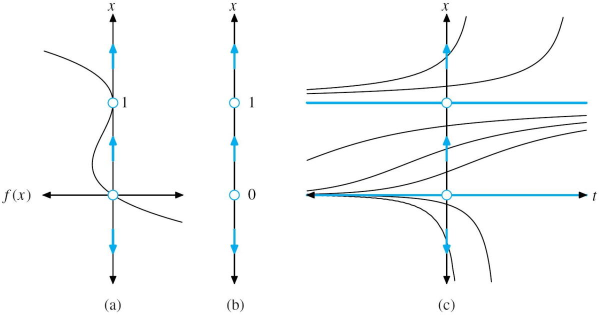
Phase Line Plots



Phase Line Plots

- Sketch graph $f(x)$ versus x
- Mark equilibrium points on x–axis
- Indicate direction of motion ($x(t)$ decreasing or increasing) by arrows
- Use this to sketch solutions

Stability Criteria



Stability Criteria

Equilibrium point x_0 is

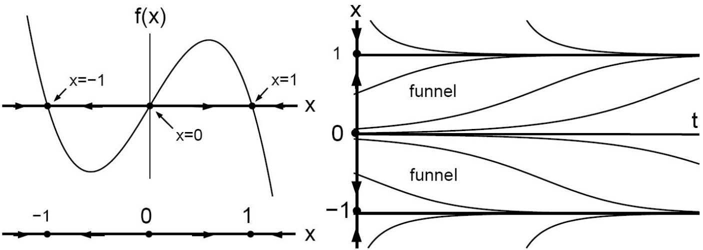
- asympt. stable if $f'(x_0) < 0$
- unstable if $f'(x_0) > 0$

If $f'(x_0) = 0$ inspect graph

Example

Ex.: $x' = x - x^3 = x(1-x)(1+x)$

- $f(x) = 0 \Rightarrow x = 0,\ 1,\ -1$
- $f'(0) = 1 \Rightarrow 0$ is unstable
- $f'(\pm 1) = -1 \Rightarrow \pm 1$ are as. stable

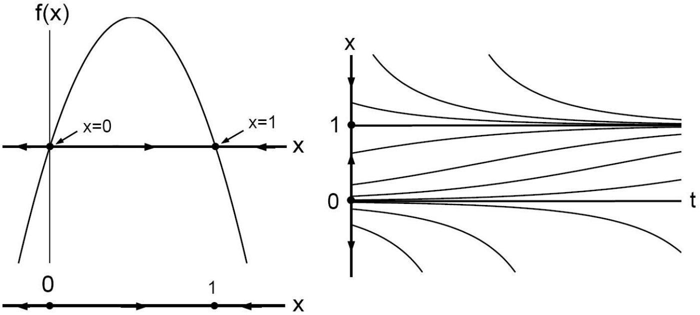


Example

Ex.: $x' = x - x^2 = x(1-x)$

Equilibria:

- $x = 0 \Rightarrow f'(0) = \ \ 1 \Rightarrow$ unstable
- $x = 1 \Rightarrow f'(1) = -1 \Rightarrow$ as. stable



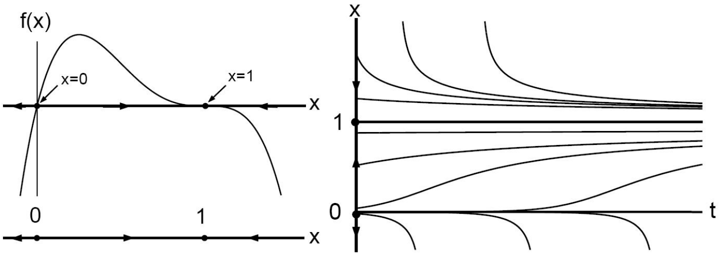
Example

Ex.: $x' = x(1-x)^3$

Equilibria:

- $x = 0 \Rightarrow f'(0) = 1 \Rightarrow$ unstable
- $x = 1 \Rightarrow f'(1) = 0 \Rightarrow$??

  Graph $\Rightarrow$ asympt. stable



Example

Ex.: $x' = -x(1-x)^2$

Equilibria:

- $x = 0 \Rightarrow f'(0) = -1 \Rightarrow$ as. stable
- $x = 1 \Rightarrow f'(1) = \ 0 \Rightarrow$??
- Inspect graph: $\Rightarrow x = 1$ is as. stable on right side, unstable on left side (semistable)