

Math 3331 Differential Equations

4.3 Linear, Homogeneous Equations with Constant Coefficients

Jiwen He

Department of Mathematics, University of Houston

`jiwenhe@math.uh.edu`
`math.uh.edu/~jiwenhe/math3331`



4.3 Linear, Homogeneous Equations with Constant Coefficients

- Definition and Key Idea
- DE and its Characteristic Equation
- Characteristic Roots and General Solution
 - Distinct Real Roots
 - Complex Roots
 - Repeated Roots
- Worked out Examples from Exercises
 - Distinct Real Roots: 2, 25
 - Complex Roots: 10
 - Repeated Roots: 18



The Key Idea

Linear, Homogeneous Equations with Constant Coefficients

$$y'' + py' + qy = 0$$

where p and q are constant.

The Key Idea

Look for a solution of the type $y(t) = e^{\lambda t}$ where λ is a constant, as yet unknown. Inserting it into the DE,

$$y'' + py' + qy = \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} = (\lambda^2 + p\lambda + q)e^{\lambda t} = 0.$$

Since $e^{\lambda t} \neq 0$, then

$$\lambda^2 + p\lambda + q = 0$$

This is called the **characteristic equation** for the DE.



Characteristic Root

DE and its Characteristic Equation

$$y'' + py' + qy = 0$$

$$\lambda^2 + p\lambda + q = 0$$

Characteristic Root

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

- two distinct real roots if $p^2 - 4q > 0$.
- two distinct complex roots if $p^2 - 4q < 0$.
- one repeated real root if $p^2 - 4q = 0$.



Distinct Real Roots

Proposition 3.3

If the characteristic equation $\lambda^2 + p\lambda + q = 0$ has two **distinct real roots** λ_1 and λ_2 , then the general solution to $y'' + py' + qy = 0$ is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

where C_1 and C_2 are arbitrary constants.

IVP

The particular solution for an initial value problem can be found by evaluating the constants C_1 and C_2 using the initial conditions.



Example 3.4

Find the general solution to the equation

$$y'' - 3y' + 2y = 0.$$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = 1$. (Ans: $y(t) = -e^{2t} + 3e^t$)

DE, its Characteristic Equation and roots

$$y'' - 3y' + 2y = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 1$$

The general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{2t} + C_2 e^t \Rightarrow y'(t) = 2C_1 e^{2t} + C_2 e^t$$

ICs: $y(0) = 2 = C_1 + C_2$ and $y'(0) = 1 = 2C_1 + C_2$ imply

$$C_1 = -1, C_2 = 3 \Rightarrow y(t) = -e^{2t} + 3e^t$$



Complex Roots

Proposition 3.20

Suppose the characteristic equation $\lambda^2 + p\lambda + q = 0$ has two **complex conjugate roots** $\lambda = a + ib$ and $\bar{\lambda} = a - ib$.

1. The functions

$$z(t) = e^{\lambda t} = e^{(a+ib)t} \quad \text{and} \quad \bar{z}(t) = e^{\bar{\lambda}t} = e^{(a-ib)t}$$

form a **complex valued** fundamental set of solutions, so the general solution the general solution to $y'' + py' + qy = 0$ is

$$y(t) = C_1 e^{\lambda t} + C_2 e^{\bar{\lambda}t} = C_1 e^{(a+ib)t} + C_2 e^{(a-ib)t}$$

where C_1 and C_2 are arbitrary **complex** constants.



Complex Roots (cont.)

Proposition 3.20 (cont.)

2. The functions

$$y_1(t) = e^{at} \cos(bt) \quad \text{and} \quad y_2(t) = e^{at} \sin(bt)$$

form a **real valued** fundamental set of solutions, so the general solution the general solution to $y'' + py' + qy = 0$ is

$$y(t) = e^{at}(A_1 \cos(bt) + A_2 \sin(bt)),$$

where A_1 and A_2 are arbitrary **real** constants.

Real and Imaginary Parts

$$z(t) = y_1(t) + iy_2(t), \quad \bar{z}(t) = y_1(t) - iy_2(t)$$

$$y_1(t) = \operatorname{Re} z(t) = \frac{1}{2}(z(t) + \bar{z}(t)), \quad y_2(t) = \operatorname{Im} z(t) = \frac{1}{2i}(z(t) - \bar{z}(t))$$

Example 3.21

Find the general solution to the equation

$$y'' + 2y' + 2y = 0.$$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = 3$. (Ans: $y(t) = e^{-t}(2 \cos t + 5 \sin t)$)

DE, its Characteristic Equation and roots

$$y'' + 2y' + 2y = 0 \Rightarrow \lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = -1 \pm i$$

The general solution is

$$y(t) = e^{at}(C_1 \cos(bt) + C_2 \sin(bt)) = e^{-t}(C_1 \cos(t) + C_2 \sin(t))$$

$$\Rightarrow y'(t) = -e^{-t}(C_1 \cos(t) + C_2 \sin(t)) + e^{-t}(-C_1 \sin(t) + C_2 \cos(t))$$

ICs: $y(0) = 2 = C_1$ and $y'(0) = 3 = -C_1 + C_2$ imply

$$C_1 = 2, C_2 = 5 \Rightarrow y(t) = e^{-t}(2 \cos(t) + 5 \sin(t))$$



Repeated Roots

Proposition 3.28

If the characteristic equation $\lambda^2 + p\lambda + q = 0$ has only one **double root** λ_1 , then the general solution to $y'' + py' + qy = 0$ is

$$y(t) = (C_1 + C_2 t)e^{\lambda_1 t}$$

where C_1 and C_2 are arbitrary constants.

IVP

The particular solution for an initial value problem can be found by evaluating the constants C_1 and C_2 using the initial conditions.



Example 3.29

Find the general solution to the equation

$$y'' - 2y' + y = 0.$$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = -1$. (Ans: $y(t) = 2e^t - 3te^t$)

DE, its Characteristic Equation and roots

$$y'' - 2y' + y = 0 \Rightarrow \lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = 1$$

The general solution is

$$\begin{aligned} y(t) &= (C_1 + C_2 t)e^{\lambda_1 t} = (C_1 + C_2 t)e^t \\ \Rightarrow y'(t) &= (C_1 + C_2 t)e^t + C_2 e^t \end{aligned}$$

ICs: $y(0) = 2 = C_1$ and $y'(0) = -1 = C_1 + C_2$ imply

$$C_1 = 2, C_2 = -3 \Rightarrow y(t) = (2 - 3t)e^t.$$



Exercise 4.3.2

Find the general solution to the equation

$$y'' + 5y' + 6y = 0.$$

DE, its Characteristic Equation and roots

$$y'' + 5y' + 6y = 0 \Rightarrow \lambda^2 + 5\lambda + 6 = 0 \Rightarrow \lambda_1 = -3, \lambda_2 = -2$$

The general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-3t} + C_2 e^{-2t}$$



Exercise 4.3.10

Find the general solution to the equation

$$y'' + 4 = 0.$$

DE, its Characteristic Equation and roots

$$y'' + 4 = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2i$$

The general solution is

$$y(t) = e^{at}(C_1 \cos(bt) + C_2 \sin(bt)) = C_1 \cos(2t) + C_2 \sin(2t)$$



Exercise 4.3.18

Find the general solution to the equation

$$y'' - 6y' + 9y = 0.$$

DE, its Characteristic Equation and roots

$$y'' - 6y' + 9y = 0 \quad \Rightarrow \quad \lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 3$$

The general solution is

$$y(t) = (C_1 + C_2 t)e^{\lambda_1 t} = (C_1 + C_2 t)e^{3t}$$



Exercise 4.3.25

Find the solution of the initial value problem

$$y'' - y' - 2y = 0, \quad y(0) = -1, \quad y'(0) = 2.$$

DE, its Characteristic Equation and roots

$$y'' - y' - 2y = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -1$$

The general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{2t} + C_2 e^{-t} \Rightarrow y'(t) = 2C_1 e^{2t} - C_2 e^{-t}$$

ICs: $y(0) = -1 = C_1 + C_2$ and $y'(0) = 2 = 2C_1 - C_2$ imply

$$C_1 = 1/3, C_2 = -4/3 \Rightarrow y(t) = \frac{1}{3}e^{2t} - \frac{4}{3}e^{-t}$$

