Math 3331 Differential Equations

5.2 Basic Properties of the Laplace Transform

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5.2 Basic Properties of the Laplace Transform

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Linearity

1. Linearity:

$$\mathcal{L}(af + bg)(s) = a\mathcal{L}(f)(s) + b\mathcal{L}(g)(s)$$

Reality

2. 'Reality':

$$f(t) \text{ real } \Rightarrow \mathcal{L}(f)(s) \text{ real}$$

Consequence: $f(t)$ complex $\Rightarrow$

$$\begin{aligned} \mathrm{Re}(\mathcal{L}(f)(s)) &= \mathcal{L}(\mathrm{Re}(f))(s) \\ \mathrm{Im}(\mathcal{L}(f)(s)) &= \mathcal{L}(\mathrm{Im}(f))(s) \end{aligned}$$

Derivatives

$$Y(s) = \mathcal{L}\{y(t)\}(s)$$

3. Derivatives:

$$\begin{aligned}
\mathcal{L}(y')(s) &= sY(s) - y(0) \\
\mathcal{L}(y'')(s) &= s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}(y^{(k)})(s) &= s^kY(s) - s^{k-1}y(0) - \\
&\quad s^{k-2}y'(0) - \cdots - y^{(k-1)}(0)
\end{aligned}$$

Proof 3

Proof 3. for $k = 1$: Use partial integration: $\int uv'\,dt = uv - \int u'v\,dt$

$$\int_0^T e^{-st} y'(t)\,dt = e^{-st}y(t)\Big|_0^T + s\int_0^T e^{-st}y(t)\,dt$$
$$= e^{-sT}y(T) - y(0)$$
$$+ s\int_0^T e^{-st}y(t)\,dt$$

For $T \to \infty$:

$$e^{-sT}y(T) \to 0, \quad \int_0^T e^{-st}y(t)\,dt \to Y(s)$$

$$\Rightarrow \mathcal{L}(y')(s) = sY(s) - y(0)$$

Multiplication by e^{ct}

$$F(s) = \mathcal{L}\{f(t)\}(s)$$

4. Multiplication by e^{ct} $(c \in \mathbb{C})$:

$$\mathcal{L}\{e^{ct} f(t)\}(s) = F(s - c)$$

Proof 4

Proof 4.:

$$\begin{aligned}\mathcal{L}\{e^{ct}f(t)\}(s) &= \int_0^\infty e^{-st}e^{ct}f(t)\,dt \\ &= \int_0^\infty e^{-(s-c)t}f(t)\,dt \\ &= F(s-c)\end{aligned}$$

Multiplication by t^k

$$F(s) = \mathcal{L}\{f(t)\}(s)$$

5. Multiplication by t^k: $(k = 0, 1, 2, \ldots)$

$$\mathcal{L}\{t^k f(t)\}(s) = (-1)^k F^{(k)}(s)$$

Proof 5

Proof 5. for $k = 1$:

$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt \;\Rightarrow$

$$F'(s) = \int_0^{\infty} (-t) f(t)\, dt = -\mathcal{L}\{t f(t)\}(s)$$

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$\mathcal{L}$-Transforms of Functions Encountered in ODEs

ODEs with constant coefficients
$\longrightarrow$ functions $t^k e^{ct}$, $k = 0, 1, 2, \ldots$

$\mathcal{L}$-Transforms of Functions Encountered in ODEs (cont.)

Property 5 $\Rightarrow$

$$\mathcal{L}\{t^k e^{ct}\}(s) = (-1)^k \frac{d^k}{ds^k}\mathcal{L}\{e^{ct}\}(s)$$

Property 4 $\Rightarrow$

$$\begin{aligned}\mathcal{L}\{e^{ct}\}(s) &= \mathcal{L}\{e^{ct}1\}(s) = \mathcal{L}\{1\}(s-c)\\ &= \frac{1}{s-c}\end{aligned}$$

$$\begin{aligned}\Rightarrow \mathcal{L}\{t^k e^{ct}\}(s) &= (-1)^k \frac{d^k}{ds^k}\frac{1}{s-c}\\ &= \frac{k!}{(s-c)^{k+1}} \qquad (1)\end{aligned}$$

$\mathcal{L}$-Transforms of Functions Encountered in ODEs (cont.)

(1) $\Rightarrow$ **Special Transforms:**

- $k = 0,\ c \in \mathbf{R} \Rightarrow \mathcal{L}\{e^{ct}\}(s) = \frac{1}{s-c}$
- $k = 0,\ c = i\omega \Rightarrow$

$$\mathcal{L}\{e^{i\omega t}\}(s) = \frac{1}{s-i\omega} = \frac{s+i\omega}{s^2+\omega^2} \Rightarrow$$

$$\mathcal{L}\{\cos\omega t\}(s) = \mathrm{Re}\left(\frac{s+i\omega}{s^2+\omega^2}\right) = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin\omega t\}(s) = \mathrm{Im}\left(\frac{s+i\omega}{s^2+\omega^2}\right) = \frac{\omega}{s^2+\omega^2}$$

$\mathcal{L}$-Transforms of Functions Encountered in ODEs (cont.)

- $k = 0,\ c = \alpha + i\beta \Rightarrow$

$$\mathcal{L}\{e^{\alpha t}e^{i\beta t}\}(s) = \frac{1}{s-\alpha-i\beta} = \frac{s-\alpha+i\beta}{(s-\alpha)^2+\beta^2}$$

$$\Rightarrow \mathcal{L}\{e^{\alpha t}\cos\beta t\}(s) = \frac{s-\alpha}{(s-\alpha)^2+\beta^2}$$

$$\mathcal{L}\{e^{\alpha t}\sin\beta t\}(s) = \frac{\beta}{(s-\alpha)^2+\beta^2}$$

Table of $\mathcal{L}$-Transforms

Table of $\mathcal{L}$–Transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$
1	$\frac{1}{s}$
t^k	$\frac{k!}{s^{k+1}}$
e^{ct}	$\frac{1}{s-c}$
$t^k e^{ct}$	$\frac{k!}{(s-c)^{k+1}}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$e^{\alpha t} \cos \beta t$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$
$e^{\alpha t} \sin \beta t$	$\frac{\beta}{(s-\alpha)^2+\beta^2}$

Exercise 5.2.3

3. Using linearity and Table 1,

$$\begin{aligned}
&\mathcal{L}\{t^2 + 4t + 5\}(s) \\
&\quad = \mathcal{L}\{t^2\}(s) + 4\,\mathcal{L}\{t\}(s) + 5\,\mathcal{L}\{1\}(s) \\
&\quad = \frac{2!}{s^3} + 4\left(\frac{1}{s^2}\right) + 5\left(\frac{1}{s}\right) \\
&\quad = \frac{2}{s^3} + \frac{4}{s^2} + \frac{5}{s} \\
&\quad = \frac{2 + 4s + 5s^2}{s^3},
\end{aligned}$$

provided $s > 0$.

Exercise 5.2.5

5. Using linearity and Table 1,

$$\begin{aligned}
&\mathcal{L}\{-2\cos t + 4\sin 3t\}(s) \\
&\quad = -2\,\mathcal{L}\{\cos t\}(s) + 4\,\mathcal{L}\{\sin 3t\}(s) \\
&\quad = -2\left(\frac{s}{s^2+1}\right) + 4\left(\frac{3}{s^2+9}\right) \\
&\quad = \frac{-2s(s^2+9) + 12(s^2+1)}{(s^2+1)(s^2+9)} \\
&\quad = \frac{-2s^3 + 12s^2 - 18s + 12}{(s^2+1)(s^2+9)},
\end{aligned}$$

provided $s > 0$.

Exercise 5.2.19

19. If $y' - 5y = e^{-2t}$, with $y(0) = 1$, then

$$\mathcal{L}\{y' - 5y\}(s) = \mathcal{L}\{e^{-2t}\}(s)$$
$$\mathcal{L}\{y'\}(s) - 5\,\mathcal{L}\{y\}(s) = \frac{1}{s+2}$$
$$s\,\mathcal{L}\{y\}(s) - y(0) - 5\,\mathcal{L}\{y\}(s) = \frac{1}{s+2}.$$

If we let $Y(s) = \mathcal{L}\{y\}(s)$, then

$$sY(s) - 1 - 5Y(s) = \frac{1}{s+2}$$
$$(s-5)Y(s) = 1 + \frac{1}{s+2}$$
$$Y(s) = \frac{1}{s-5} + \frac{1}{(s-5)(s+2)}$$
$$Y(s) = \frac{(s+2)+1}{(s-5)(s+2)}$$
$$Y(s) = \frac{s+3}{(s-5)(s+2)}.$$

Exercise 5.2.22

22. If

$$y'' + y = \sin 4t, \quad y(0) = 0, \quad y'(0) = 1,$$

then, letting $Y(s) = \mathcal{L}(y)(s)$,

$$\begin{aligned} s^2 \mathcal{L}(y)(s) - sy(0) - y'(0) + \mathcal{L}(y)(s) &= \frac{4}{s^2 + 4^2} \\ s^2 Y(s) - 1 + Y(s) &= \frac{4}{s^2 + 16}. \end{aligned}$$

Solving for $Y(s)$,

$$\begin{aligned} (s^2 + 1)Y(s) &= 1 + \frac{4}{s^2 + 16} \\ (s^2 + 1)Y(s) &= \frac{s^2 + 20}{s^2 + 16} \\ Y(s) &= \frac{s^2 + 20}{(s^2 + 1)(s^2 + 16)}. \end{aligned}$$

Exercise 5.2.39

39. If $y'' + y' + 2y = e^{-t}\cos 2t$, with $y(0) = 1$ and $y'(0) = -1$, then with $Y(s) = \mathcal{L}\{y\}(s)$,

$$\begin{aligned}
&\mathcal{L}\{y'' + y' + 2y\}(s) \\
&\quad = s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0) \\
&\qquad + s\,\mathcal{L}\{y\}(s) - y(0) + 2\,\mathcal{L}\{y\}(s) \\
&\quad = s^2Y(s) - s + 1 + sY(s) - 1 + 2Y(s) \\
&\quad = (s^2 + s + 2)Y(s) - s.
\end{aligned}$$

Because the transform of $f(t) = \cos 2t$ is $F(s) = s/(s^2+4)$, the transform of $e^{-t}\cos 2t$ is

$$F(s+1) = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}.$$

Equating,

$$(s^2+s+2)Y(s) - s = \frac{s+1}{s^2+2s+5}.$$

Solving for Y

$$\begin{aligned}
Y(s) &= \frac{s}{s^2+s+2} + \frac{s+1}{(s^2+s+2)(s^2+2s+5)} \\
&= \frac{s(s^2+2s+5)+s+1}{(s^2+s+2)(s^2+2s+5)} \\
&= \frac{s^3+2s^2+6s+1}{(s^2+s+2)(s^2+2s+5)}.
\end{aligned}$$