Math 3331 Differential Equations

6.2 Runge-Kutta Methods (RKM)

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6.2 Runge-Kutta Methods (RKM)

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2nd Order RKM: Improved Euler Method

Failure of Euler Method: Only slope on left end of interval $[t, t+h]$ is used.

Improvement: Given $t, y(t)$,

- compute slope at t

$$s_l = f(t, y(t))$$

- find slope at $t+h$ via EM

$$\begin{aligned} y_E &= y(t) + hs_l \\ s_r &= f(t+h, y_E) \end{aligned}$$

- approximate $y(t+h)$ via average slope

$$y(t+h) \approx y(t) + h(s_l + s_r)/2$$

2nd Order RKM: Iteration Scheme

2nd Order RKM

Iteration Scheme

Start: y_0, t_0

For $k = 0$ to $k = N$:

$$\begin{aligned} t_{k+1} &= t_k + h \\ s_l &= f(t_k, y_k) \\ s_r &= f(t_{k+1}, y_k + h s_l) \\ y_{k+1} &= y_k + h(s_l + s_r)/2 \end{aligned}$$

Example

Ex. Approximate the solution to

$$y' = t - y,\ y(0) = 0.5$$

in $0 \le t \le 1$ using $h = 0.25$.

Start: $y_0 = 0.5,\ t_0 = 0$

$$\begin{aligned} t_1 &= 0.25 \\ s_l &= t_0 - y_0 = -0.5 \\ s_r &= t_1 - (y_0 + hs_l) = -0.125 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + h(s_l + s_r)/2 = 0.4219 \\ t_2 &= 0.5 \\ s_l &= t_1 - y_1 = -0.1719 \\ s_r &= t_2 - (y_1 + hs_l) = 0.1211 \\ y_2 &= y_1 + h(s_l + s_r)/2 = 0.4155 \end{aligned}$$

$$\begin{aligned} t_3 &= 0.75 \\ s_l &= t_2 - y_2 = 0.0845 \\ s_r &= t_3 - (y_2 + hs_l) = 0.3134 \\ y_3 &= y_2 + h(s_l + s_r)/2 = 0.4653 \\ t_4 &= 1 \\ s_l &= t_3 - y_3 = 0.0845 \\ s_r &= t_4 - (y_3 + hs_l) = 0.3134 \\ y_4 &= y_3 + h(s_l + s_r)/2 = 0.4653 \end{aligned}$$

Errors in 2nd Order RKM: Second Order

Ex.: $y' = t - y, \quad y(0) = 0.5$
Approximate $y(1)$ for stepsizes

$$h = 1/m, \quad m = 1, 2, 4, 8, 16, 32$$

Exact Value: $y(1) = 0.551819$

Error: $E(h) = |y(1) - y_m|$

h	y_m	$E(h)$
1	0.75	0.198181
1/2	0.585938	0.034118
1/4	0.558794	0.006974
1/8	0.553400	0.001581
1/16	0.552196	0.000377
1/32	0.551911	0.000092

$$E(h/2) \approx E(h)/4 \;\Rightarrow\; E(h) \approx C\,h^2$$

Theorem: There $\exists\ C > 0$ s.t.

$$E(h) \le C\,h^2$$

(2nd order RKM is second order method)

4th Order RKM: Basic Idea

4th Order RKM

Idea: Given t and $y = y(t)$, compute slopes s_1, s_2, s_3, s_4 at four carefully chosen points s.t. error is minimized.

Approximation:

$$y(t+h) \approx y + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

4th Order RKM: Iteration Scheme

4th Order RKM

Iteration k → k + 1:

$$
\begin{aligned}
s_1 &= f(t_k, y_k) \\
s_2 &= f(t_k + h/2, y_k + hs_1/2) \\
s_3 &= f(t_k + h/2, y_k + hs_2/2) \\
s_4 &= f(t_k + h, y_k + hs_3) \\
y_{k+1} &= y_k + \\
&\quad \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4) \\
t_{k+1} &= t_k + h
\end{aligned}
$$

Errors in 4th Order RKM: Fourth Order

Ex.: $y' = t - y,\ y(0) = 0.5,\ y(1) \approx y_m$

$m = 1, 2, 4, 8, 16, 32,\ h = 1/m$

Exact Value: $y(1) = 0.551819162$

Error: $E(h) = |y(1) - y_m|$

h	y_m	$E(h)$
1	0.5625	0.010680838
1/2	0.552256266	0.000437105
1/4	0.551841299	0.000022137
1/8	0.551820408	0.000001246
1/16	0.551819236	0.000000074
1/32	0.551819166	0.000000005

$$E(h/2) \approx E(h)/16 \ \Rightarrow\ E(h) \approx C\,h^4$$

Theorem: There $\exists\ C > 0$ s.t.

$$E(h) \le C\,h^4$$

(4th order RKM is fourth order method)