Math 3331 Differential Equations

8.1 Introduction to Systems

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8.1 Introduction to Systems Definitions and Examples

- Definitions
 - System of First Order ODEs
 - IVP: Existence and Uniqueness of Solution
- Examples
- Reduction of Higher Order Equations
- Worked out Examples from Exercises:
 - 1, 2, 7





System of First Order ODEs

System of 1st order ODEs:

$$x'_1 = f_1(t, x_1, \dots, x_n)$$

$$\vdots$$

$$x'_n = f_n(t, x_1, \dots, x_n)$$

Vector notation:

$$\mathbf{x} = [x_1, \dots, x_n]^T$$

$$\mathbf{f} = [f_1, \dots, f_n]^T$$

$$\mathbf{x}' = [x'_1, \dots, x'_n]^T$$

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$$
(1)

n: dimension of system n = 2: planar system

- (1) is autonomous if f does not depend on t
- (1) is non-autonomous if **f** depends on *t*





IVP: Existence and Uniqueness of Solution

Initial Value Problem:

$$x' = f(t, x), x(t_0) = x_0$$
 (2)

Thm.: If f is continuous in a region R and has continuous partial derivatives $\partial f_i/\partial x_j$ in R, (2) has a unique solution in R.





Example 1

Ex.:
$$x'_1 = -ax_1x_2$$

 $x'_2 = ax_1x_2 - bx_2$
 $x'_3 = bx_2$

$$\mathbf{x} = [x_1, x_2, x_3]^T$$

$$\mathbf{f}(\mathbf{x}) = [-ax_1x_2, ax_1x_2 - bx_2, bx_2]^T$$

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \text{ is } 3d \text{ autonomous system}$$





Example 2

Ex.:
$$x' = v$$
 $v' = -x - 0.2v + 2\cos t$

is 2d non-autonomous system





Reduction of Higher Order Equations

Thm. Any system of higher order ODEs in explicit form can be transformed to a (larger) system of first order ODEs.





Example 3

$$\mathbf{Ex.:} \quad x''' + xx'' = \cos t \tag{3}$$

Set
$$x_1 = x$$
, $x_2 = x'$, $x_3 = x''$
 $\Rightarrow x'_1 = x' = x_2$
 $x'_2 = x'' = x_3$
 $x'_3 = x''' = -xx'' + \cos t$
 $= -x_1x_3 + \cos t$

Given a solution x(t) of (3) \Rightarrow $[x(t), x'(t), x''(t)]^T$ is solution of (4)

Hence equivalent system:

$$x'_1 = x_2$$

 $x'_2 = x_3$
 $x'_3 = -x_1x_3 + \cos t$ (4)

Conversely: Given a solution $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T$ of (4) \Rightarrow $x(t) = x_1(t)$ is a solution of (3)





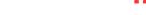
General Higher Order ODEs

General Higher Order ODEs:

nth order ODE in explicit form:

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$





Exercise 8.1.1

Ex. 1: Is the system autonomous? What is the dimension?

$$x'=v$$
 $v'=-x-0.02v+2\cos t$ is non-autonomous (cos t). Dimension: 2





Exercise 8.1.2

Ex. 2: Same questions as in Ex. 1

$$\theta' = \omega$$
 $\omega' = -(g/L)\sin\theta + (k/m)\omega$ is autonomous. Dimension: 2





Exercise 8.1.7

Ex. 7: Show that given functions are solutions of initial value problem

$$\text{IVP: } \left\{ \begin{array}{lll} x' & = & -4x + 6y \\ y' & = & -3x + 5y \end{array} \right\}, \ \left\{ \begin{array}{lll} x(0) = 0 \\ y(0) = 1 \end{array} \right\}; \ \text{functions} \ \left\{ \begin{array}{lll} x(t) & = & 2e^{2t} - 2e^{-t} \\ y(t) & = & -e^{-t} + 2e^{2t} \end{array} \right\}$$

$$x'(t) = 4e^{2t} + 2e^{-t}, -4x(t) + 6y(t) = -4(2e^{2t} - 2e^{-t}) + 6(-e^{-t} + 2e^{2t}) = 4e^{2t} + 2e^{-t}$$
$$y'(t) = e^{-t} + 4e^{2t}, -3x(t) + 5y(t) = -3(2e^{2t} - 2e^{-t}) + 5(-e^{-t} + 2e^{2t}) = e^{-t} + 4e^{2t}$$
$$IC: x(0) = 0, y(0) = 1,$$

hence x(t), y(t) are solutions of IVP



