Math 3331 Differential Equations

8.1 Introduction to Systems

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/∼jiwenhe/math3331

8.1 Introduction to Systems
Definitions and Examples

- Definitions
  - System of First Order ODEs
  - IVP: Existence and Uniqueness of Solution
- Examples
- Reduction of Higher Order Equations
- Worked out Examples from Exercises:
  - 1, 2, 7

System of First Order ODEs

System of 1st order ODEs:

$$\begin{aligned} x_1' &= f_1(t, x_1, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, \ldots, x_n) \end{aligned}$$

Vector notation:

$$\begin{aligned} \mathbf{x} &= [x_1, \ldots, x_n]^T \\ \mathbf{f} &= [f_1, \ldots, f_n]^T \\ \mathbf{x}' &= [x_1', \ldots, x_n']^T \end{aligned}$$

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \qquad (1)$$

n: dimension of system

$n = 2$: planar system

- (1) is autonomous if $\mathbf{f}$ does not depend on t
- (1) is non-autonomous if $\mathbf{f}$ depends on t

IVP: Existence and Uniqueness of Solution

Initial Value Problem:

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \qquad (2)$$

Thm.: If f is continuous in a region R and has continuous partial derivatives $\partial f_i / \partial x_j$ in R, (2) has a unique solution in R.

Example 1

Ex.:
$$\begin{aligned} x_1' &= -ax_1x_2 \\ x_2' &= ax_1x_2 - bx_2 \\ x_3' &= bx_2 \end{aligned}$$

$$\begin{aligned} \mathbf{x} &= [x_1, x_2, x_3]^T \\ \mathbf{f}(\mathbf{x}) &= [-ax_1x_2, ax_1x_2 - bx_2, bx_2]^T \end{aligned}$$

$\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is 3d autonomous system

Example 2

Ex.:

$$\begin{aligned} x' &= v \\ v' &= -x - 0.2v + 2\cos t \end{aligned}$$

is $2d$ non-autonomous system

Reduction of Higher Order Equations

Thm. Any system of higher order ODEs in explicit form can be transformed to a (larger) system of first order ODEs.

Example 3

Ex.: $x''' + xx'' = \cos t$ (3)

Set $x_1 = x,\ x_2 = x',\ x_3 = x''$

$$\Rightarrow \begin{aligned} x_1' &= x' = x_2 \\ x_2' &= x'' = x_3 \\ x_3' &= x''' = -xx'' + \cos t \\ &= -x_1x_3 + \cos t \end{aligned}$$

Given a solution $x(t)$ of (3) $\Rightarrow$ $[x(t), x'(t), x''(t)]^T$ is solution of (4)

Hence equivalent system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -x_1x_3 + \cos t \end{aligned} \qquad (4)$$

Conversely: Given a solution $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T$ of (4) $\Rightarrow$ $x(t) = x_1(t)$ is a solution of (3)

General Higher Order ODEs

General Higher Order ODEs:

nth order ODE in explicit form:

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)})$$

Set $x_1 = x$, $x_2 = x'$, ... , $x_n = x^{(n-1)}$

$$\Rightarrow \begin{aligned} x_1' &= x' = x_2 \\ x_2' &= x'' = x_3 \\ &\vdots \\ x_{n-1}' &= x^{(n-1)} = x_n \\ x_n' &= x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}) \\ &= f(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

$\Rightarrow$ equivalent system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= f(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

Exercise 8.1.1

Ex. 1: Is the system autonomous? What is the dimension?

$$\left.\begin{aligned} x' &= v \\ v' &= -x - 0.02v + 2\cos t \end{aligned}\right\}$$ is non-autonomous ($\cos t$). Dimension: 2

Exercise 8.1.2

Ex. 2: Same questions as in Ex. 1

$$\left.\begin{array}{rcl} \theta' & = & \omega \\ \omega' & = & -(g/L)\sin\theta + (k/m)\omega \end{array}\right\} \text{ is autonomous. Dimension: 2}$$

Exercise 8.1.7

Ex. 7: Show that given functions are solutions of initial value problem

$$\text{IVP: } \left\{ \begin{array}{rcl} x' &=& -4x+6y \\ y' &=& -3x+5y \end{array} \right\}, \left\{ \begin{array}{c} x(0)=0 \\ y(0)=1 \end{array} \right\}; \text{ functions } \left\{ \begin{array}{rcl} x(t) &=& 2e^{2t}-2e^{-t} \\ y(t) &=& -e^{-t}+2e^{2t} \end{array} \right\}$$

$$x'(t) = 4e^{2t}+2e^{-t},\ -4x(t)+6y(t) = -4(2e^{2t}-2e^{-t})+6(-e^{-t}+2e^{2t}) = 4e^{2t}+2e^{-t}$$

$$y'(t) = e^{-t}+4e^{2t},\ -3x(t)+5y(t) = -3(2e^{2t}-2e^{-t})+5(-e^{-t}+2e^{2t}) = e^{-t}+4e^{2t}$$

$$\text{IC: } x(0)=0,\ y(0)=1,$$

hence $x(t), y(t)$ are solutions of IVP