Math 3331 Differential Equations

8.4 Linear Systems

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8.4 Linear Systems

- Linear Systems: General Form
- Linear Systems: Matrix-Vector Notation
- Initial Value Problem
- Examples
- Worked out Examples from Exercises:
    - 13, 14

Linear Systems: General Form

General Form:

$$
\begin{aligned}
x_1' &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t) \\
x_2' &= a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n + f_2(t) \\
\vdots &\quad \vdots \quad \vdots \\
x_n' &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}
$$

- $a_{ij}(t), f_i(t)$: known functions on interval I: $\alpha < t < \beta$

Linear Systems: Matrix-Vector Notation

Matrix-vector notation:

$$\begin{aligned} \mathbf{x} &= [x_1, \ldots, x_n]^T \\ \mathbf{f}(t) &= [f_1(t), \ldots, f_n(t)]^T \\ A(t) &= [a_{ij}(t)]_{nn} \end{aligned}$$

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t) \qquad (1)$$

- (1) is **homogeneous** if
  $$\mathbf{f}(t) = \mathbf{0}$$
- (1) is **nonhomogeneous** if
  $$\mathbf{f}(t) \neq \mathbf{0}$$
- (1) has constant coefficients if $a_{ij}(t) = a_{ij}$ are constants

Initial Value Problem

Initial Value Problem:

$$\left.\begin{aligned} \mathbf{x}' &= A(t)\mathbf{x} + \mathbf{f}(t) \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \end{aligned}\right\} \quad (2)$$

Thm.: If $a_{ij}(t)$, $f_i(t)$ are continuous on I and $t_0 \in I$, then (2) has a unique solution on I.

Example 1

Ex.: $\left\{ \begin{array}{rcl} x' & = & 3x - 5y \\ y' & = & -2x \end{array} \right\} \Rightarrow$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is hom., constant coefficients.

Example 2

Ex.: $\left\{ \begin{array}{ccc} u' & = & \cos(t)v \\ v' & = & u + \sin t \end{array} \right\}$

is nonhom., non-constant coefs.:

$$A(t) = \begin{bmatrix} 0 & \cos t \\ 1 & 0 \end{bmatrix}, \mathbf{f}(t) = \begin{bmatrix} 0 \\ \sin t \end{bmatrix}$$

Example 3

Ex.: $x' = xy$, $y' = x$ is nonlinear

Exercise 8.4.13

Ex. 8.4.13: If possible, place system in form (1), if not possible explain why.

$$\left\{ \begin{array}{rcl} x_1' & = & -2x_1 + x_2^2 \\ x_2' & = & 3x_1 - x_2 \end{array} \right\}$$ cannot be placed because it is nonlinear.

Exercise 8.4.14

Ex. 8.4.14: Same as Ex. 8.4.13

$$\left\{ \begin{array}{rcl} x_1' & = & -2x_1 + 3tx_2 + \cos t \\ tx_2' & = & x_1 - 4tx_2 + \sin t \end{array} \right\} \rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 3t \\ 1/t & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \cos t \\ (\sin t)/t \end{bmatrix}$$