Math 3331 Differential Equations

9.2 Planar Systems

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9.2 Planar Systems

- Planar Systems
- Solutions of 2d Systems for Distinct Real Eigenvalues
- Examples
- Worked out Examples from Exercises:
    - 3, 9

Planar Systems

2d Systems: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
(Sec. 9.2-4)

$$\begin{aligned} p(\lambda) &= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad-bc) \end{aligned}$$

Set $T = a + d$ (trace of A)
$D = ad - bc$ (det(A))

$\Rightarrow p(\lambda) = \lambda^2 - T\lambda + D$

Roots of $p(\lambda)$:

$$\lambda_{1,2} = \left(T \pm \sqrt{T^2 - 4D}\right)/2$$

Roots are real and distinct if

$$T^2 - 4D > 0$$

Solutions of 2d Systems for Distinct Real Eigenvalues

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \left\{ \begin{array}{lcl} T & = & a+d \\ D & = & ad-bc \end{array} \right\}$$

$$p(\lambda) = \lambda^2 - T\lambda + D$$

Assume $T^2 - 4D > 0$

$\Rightarrow$ A has two distinct real eigenvalues $\lambda_{1,2}$

Let $\mathbf{v}_1 \neq \mathbf{0}$ be in null$(A - \lambda_1 I)$
$\mathbf{v}_2 \neq \mathbf{0}$ be in null$(A - \lambda_2 I)$

$\mathbf{v}_1, \mathbf{v}_2$ are linearly independent

$\Rightarrow$ Fundamental Solution Set:

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1, \ \ \mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$$

Fundamental Matrix:

$$X(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t)]$$

General Solution:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = X(t)\mathbf{c}$$

Example

Ex.: $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} \Rightarrow T = 1,\ D = -2$

$\Rightarrow\ p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$

$\Rightarrow$ Eigenvalues: $\lambda_1 = 2,\ \lambda_2 = -1$

$$A - 2I = \begin{bmatrix} -6 & 6 \\ -3 & 3 \end{bmatrix}, (A - 2I)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A + I = \begin{bmatrix} -3 & 6 \\ -3 & 6 \end{bmatrix}, (A + I)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ eigenvectors $\left\{ \begin{array}{l} \mathbf{v}_1 = [1, 1]^T \\ \mathbf{v}_2 = [2, 1]^T \end{array} \right\}$

$\Rightarrow\ \mathbf{x}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x}_2(t) = e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

are a fundamental set of solutions.

Fundamental matrix:

$$X(t) = \begin{bmatrix} e^{2t} & 2e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}$$

General Solution:

$$\mathbf{x}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = X(t)\mathbf{c}$$

Exercise 9.2.3

Ex. 9.2.3: Find general solution of $\mathbf{y}' = A\mathbf{y}$ for $A = \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix}$

$T = -7$, $D = 12 \Rightarrow T^2 - 4D = 1 \Rightarrow$ eigenvalues $\lambda_{1,2} = -7/2 \pm 1/2$
$\Rightarrow \lambda_1 = -3$, $\lambda_2 = -4$. Find eigenvectors:

$$A + 3I = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad A + 4I = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Fundamental set of solutions:

$$\mathbf{y}_1(t) = e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{y}_2(t) = e^{-4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

General solution:

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) = \begin{bmatrix} e^{-3t} & e^{-4t} \\ 2e^{-3t} & e^{-4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Exercise 9.2.9

Ex. 9.2.9: Find solution of system of Ex. 3 for IC $\mathbf{y}(0) = [0, -1]^T$

Match c_1, c_2 to IC:

$$\mathbf{y}(0) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{y}(t) = -e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-4t} - e^{-3t} \\ e^{-4t} - 2e^{-3t} \end{bmatrix}$$