Math 3331 Differential Equations

9.7 Qualitative Analysis of Linear Systems

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9.7 Qualitative Analysis of Linear Systems

- Stability
  - Definitions
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  - Theorems
- Stability of 2D Systems
- Worked out Examples from Exercises:
  - 1, 3, 5, 7

Stability: Definitions

$$\mathbf{x}' = A\mathbf{x}, \quad A: n \times n \qquad (1)$$

$\mathbf{x}(t) = \mathbf{0}$ is equilibrium solution

Consider general system:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \qquad (2)$$

Assume equilibrium $\mathbf{x}(t) = \mathbf{x}_0$:

$$\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$$

Def.:

- $\mathbf{x}_0$ is stable if for any $\epsilon > 0$ there is a $\delta > 0$ s.t. $|\mathbf{x}(t) - \mathbf{x}_0| < \epsilon$ for all $t > 0$ whenever $|\mathbf{x}(0) - \mathbf{x}_0| < \delta$. (Solutions that start close to $\mathbf{x}_0$ remain close.)

Def.:

- $\mathbf{x}_0$ is unstable if it is not stable. (There are solutions starting arbitrarily close to $\mathbf{x}_0$ that move 'far away' from $\mathbf{x}_0$.)
- $\mathbf{x}_0$ is asymptotically stable if $\mathbf{x}_0$ is stable and there is $\eta > 0$ s.t. $\mathbf{x}(t) \to \mathbf{x}_0$ for $t \to \infty$ whenever $|\mathbf{x}(0) - \mathbf{x}_0| < \eta$.

Def.:

- An asymptotically stable equilibrium $\mathbf{x}_0$ of (2) is a sink.
- An equilibrium $\mathbf{x}_0$ of (2) is a source if every solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0|$ arbitrarily small eventually moves 'far away' from $\mathbf{x}_0$ when t increases.

Stability: Examples

$$\mathbf{x}' = A\mathbf{x}, \quad A : n \times n \qquad (1)$$

$\mathbf{x}(t) = \mathbf{0}$ is equilibrium solution

Examples:

Let A be 2×2.

The equilibrium $\mathbf{x}_0 = \mathbf{0}$ of (1) is

- a sink if the phase portrait is a nodal or spiral sink
- a source if the phase portrait is a nodal or spiral source
- unstable if the phase portrait is a saddle
- stable but not asymptotically stable if the phase portrait is a center or stable saddle-node.

Stability: Theorems

Thm.: Let A be $n \times n$

1. If $\text{Re}(\lambda) < 0$ for all eigenvalues of A ($\lambda < 0$ if λ is real), then $\mathbf{x}(t) \to \mathbf{0}$ for $t \to \infty$ for any solution $\mathbf{x}(t)$ of (1).
   ($\mathbf{0}$ is a sink)

2. If there is an eigenvalue λ of A with $\text{Re}(\lambda) > 0$ ($\lambda > 0$ if λ is real), then there are solutions $\mathbf{x}(t)$ of (1) with $|\mathbf{x}(0)|$ arbitrarily small that get arbitrarily large when t increases.
   ($\mathbf{0}$ is unstable)

3. If $\text{Re}(\lambda) > 0$ for all eigenvalues λ of A, then every solution $\mathbf{x}(t)$ of (1) with $\mathbf{x}(0) \neq \mathbf{0}$ gets arbitrarily large when t increases.
   ($\mathbf{0}$ is a source)

4. If $\text{Re}(\lambda) \leq 0$ for all eigenvalues λ of A, and for any eigenvalue with $\text{Re}(\lambda) = 0$ every generalized eigenvector is an eigenvector, then **$\mathbf{0}$ is stable**. *(Ex.: stable saddle-nodes, centers)*

Stability of 2D Systems

For $\mathbf{n = 2}$:

- $D > 0$, $T < 0 \Rightarrow$ sink
- $D > 0$, $T > 0 \Rightarrow$ source
- $D < 0 \Rightarrow$ saddle $\Rightarrow$ unstable but not source

Ex.: $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \mathbf{x}(t) = \begin{bmatrix} x_0 \\ e^{-t} y_0 \end{bmatrix}$

$\lambda = 0 \leftrightarrow \mathbf{v} = [1, 0]^T$: $\mathbf{0}$ is stable

Ex.: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} T = 0 \\ D = 0 \end{Bmatrix} \Rightarrow p(\lambda) = \lambda^2$

$\lambda = 0 \leftrightarrow \mathbf{v} = [1, 0]^T$

$A^2 = 0 \Rightarrow \begin{cases} \text{every vector is} \\ \text{generalized eigenvector} \end{cases}$

Solution:

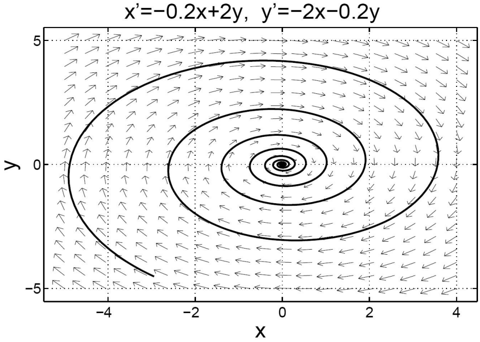
$\mathbf{x}(t) = (I + At)\mathbf{x}_0 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 + t x_0 \end{bmatrix}$

$\Rightarrow \mathbf{0}$ is unstable (but not a source)

Exercise 9.7.1

Ex. 1: Classify $\mathbf{0}$ as unstable equilibrium, stable equilibrium, sink or source of $\mathbf{x}' = A\mathbf{x}$ for the given A. Verify the classification through a phase portrait.

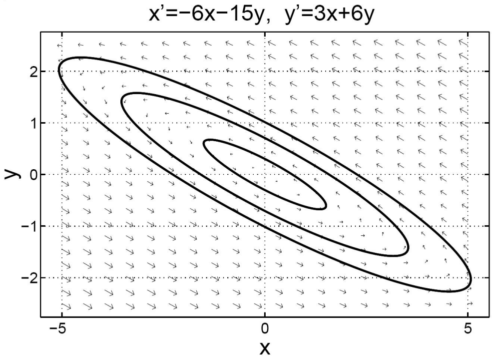
$$A = \begin{bmatrix} -0.2 & 2 \\ -2 & -0.2 \end{bmatrix}: \ D = 4.04 > 0, \ T = -0.4 < 0 \Rightarrow \text{sink (spiral sink)}$$



Exercise 9.7.3

Ex. 3: Same as Ex. 1 for $A = \begin{bmatrix} -6 & -15 \\ 3 & 6 \end{bmatrix}$

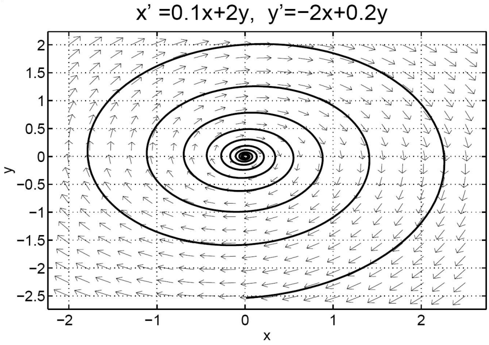
$D = 9$, $T = 0 \Rightarrow$ center $\Rightarrow$ stable (but not sink)



Exercise 9.7.5

Ex. 5: Same as Ex. 1 for $A = \begin{bmatrix} 0.1 & 2 \\ -2 & 0.1 \end{bmatrix}$.

$D = 4.04$, $T = 0.2 \Rightarrow$ source (phase portrait: spiral source)



Exercise 9.7.7

Ex. 7: Same as Ex. 1 for $A = \begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix}$.

$D = -2 \Rightarrow$ saddle $\Rightarrow$ unstable (but not source)