Expetation

* Definition. Let X be a discrete rv with set of possible values D and pmf $p(x)$. The expectation of X, denoted by $E(X)$ is
or (μ)

$$E(X) = \sum_{x \in D} x\, p(x)$$

As center of gravity of $p(x)$ if we present the distribution by the histogram, with bars of uniform unit density

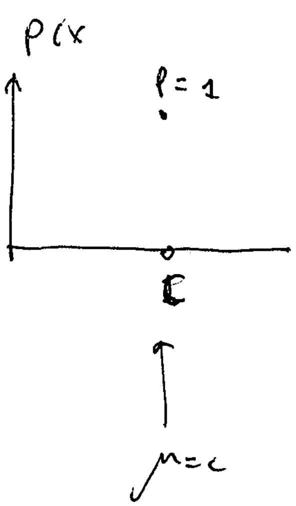
* Remark: * $E(X)$ is also known as the expected value of X, or the mean of X, or the first moment of X

  * We assume that the summation converges absolutely. i.e,

$$\sum_{x \in D} |x|\, p(x) < \infty$$

* Examples * Trivial rv. Let X be constant, i.e.

$$X(\omega) = c \qquad \forall \omega \in \Omega$$



Then

$$p(x) = \begin{cases} 1, & x = c \\ 0 & x \neq c \end{cases}$$

and

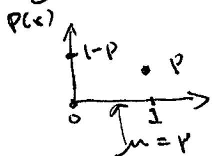
$$E(X) = c$$

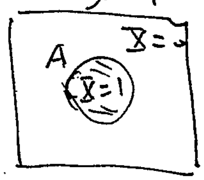
- Bernoulli r.v./indicator r.v. Let X can take only the values 0 or 1, then X is said to be a Bernoulli r.v./indicator r.v. If we define the event on which $X = 1$,

$$A = \{\omega : X(\omega) = 1\}$$

then X is said to be the indicator of A.

Ex: Flip a fair coin with
* $p(0) = p(1) = \frac{1}{2}$
* $\mu = \frac{1}{2}$





$P(A) = p$

Then

$$p(x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \\ 0 & x \neq 0, 1 \end{cases}$$

and

$$E(X) = p \quad (= 1 \times p + 0 \times (1-p))$$

- Uniform r.v. Let X be uniform on the integers $\{1, 2, \dots, n\}$. Then

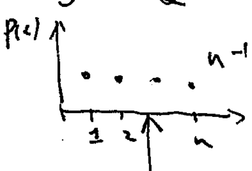
$\Sigma p(x) = 1$,

$$p(x) = \begin{cases} n^{-1}, & 1 \le x \le n, \ x \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$

and

$$E(X) = n^{-1} \sum_{j=1}^{n} j = \frac{1}{2}(n+1) = \text{the average of } \{1, .., n\}$$

Ex: Roll a fair die with
* $p(j) = \frac{1}{6}$, $1 \le j \le 6$
* $\mu = \frac{7}{2}$

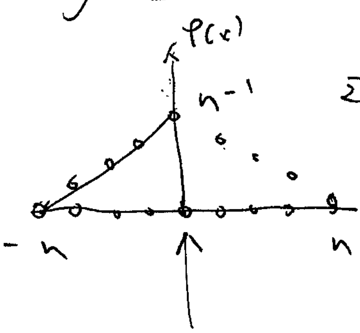


$\mu = \frac{1}{2}(n+1)$

- Triangular r.v. In this case, we have

$$\Sigma p(j) = 1, \quad p(j) = n^{-2} \min\{n+j, n-j\}$$

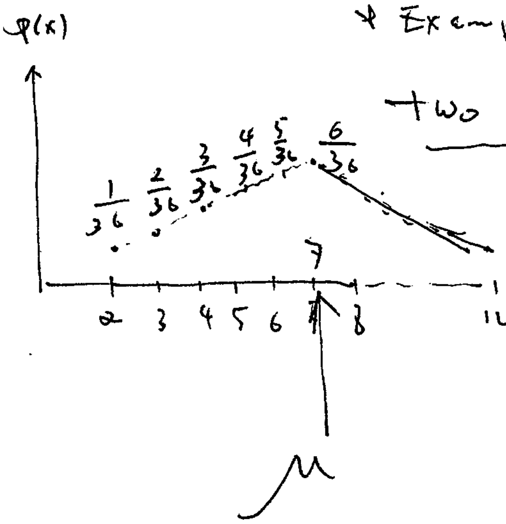
$$= \begin{cases} n^{-2}(n-j), & 0 \le j \le n \\ n^{-2}(n+j) & -n \le j \le 0 \end{cases}$$



$\mu = 0$

and

$$E(X) = 0 \quad \left(= \sum_{j=0}^{n} j\, n^{-2}(n-j) + \sum_{j=-n}^{0} j\, n^{-2}(n+j) \right)$$



* Example. Let X = the sum of the scores of two fair dice. Then

$$p(j) = 6^{-2} \min\{j-1,\ 13-j\}, \quad 2 \le j \le 12$$

and

$$E(X) = 7$$

* Geometric mean. Let X be a geometric rv with parameter p. Then

Ex: A sequence of independent Bernoulli trials with probability p to win.

$\sum_{j=1}^{\infty} p(j) = 1$,

$$p(j) = p(1-p)^{j-1}, \quad j = 1, 2, 3, \dots$$

X = nb of trials needed until the first win.

and

$$E(X) = \sum_{j=1}^{\infty} j\, p(1-p)^{j-1} = p \sum_{j=1}^{\infty} \left[-\frac{d}{dp}(1-p)^{j} \right]$$

$$= p^{-1}$$

* Tail sum. When $X \ge 0$ and X is integer valued, show that

* geometric mean

$$E(X) = \sum_{j=0}^{\infty} (1-p)^{j} = \frac{1}{p}$$

$$E(X) = \sum_{j=0}^{\infty} \{1 - F(j)\}$$

Proof.

$$E(X) = \sum_{j=1}^{\infty} j\, p(j) = \begin{array}{l} p(1) \\ +\ p(2) + p(2) \\ +\ p(3) + p(3) + p(3) \\ +\ \cdots \end{array}$$

$$= \sum_{j=1}^{\infty} p(j) + \sum_{j=2}^{\infty} p(j) + \sum_{j=3}^{\infty} p(j) + \cdots$$

$$= [1 - F(0)] + [1 - F(1)] + [1 - F(2)] + \cdots = \sum_{j=0}^{\infty} \{1 - F(j)\}$$

Expectation of functions

Theorem: Let X and Y be discrete, with

$$Y = g(X).$$

Then

$$E(Y) = \sum_{x \in D} g(x)\,p(x)$$

Proof: ~~For~~ For some fixed y,

$$\sum_{\substack{x \in D \\ g(x)=y}} g(x)\,p(x) = \sum_{\substack{x \in D \\ g(x)=y}} y\,p(x) = y \sum_{\substack{x \in D \\ g(x)=y}} p(x)$$

Given the distribution of X, $p(x)$, Y has distribution

$$p_Y(y) = \sum_{\substack{x \in D \\ g(x)=y}} p(x)$$

Then

$$E(Y) = \sum_y y\,p_Y(y) = \sum_y \left\{ y \sum_{\substack{x \in D \\ g(x)=y}} p(x) \right\}$$

$$= \sum_y \sum_{\substack{x \in D \\ g(x)=y}} g(x)\,p(x)$$

$$= \sum_{x \in D} g(x)\,p(x)$$

Remark. 1/ We do not need to find the distribution of Y in order to find its mean.

2/ It is not true in general that

$$E[g(X)] = g[E(X)].$$

Corollary: linear transformation

$$E[a\underline{X}+b] = a\,E[X] + b$$

Proof.

$$E[a\underline{X}+b] = \sum_x (ax+b)p(x) = a\sum_x x\,p(x) + b\sum_x p(x)$$

$$= a\,E[X] + b$$

Corollary: linearity of expectation

$$E[g(\underline{X}) + h(\underline{X})] = E[g(\underline{X})] + E[h(\underline{X})]$$

Proof:

$$E[g(\underline{X}) + h(\underline{X})] = \sum_x (g(x) + h(x))\,p(x)$$

$$= \sum_x g(x)p(x) + \sum_x h(x)p(x)$$

$$= E[g(\underline{X})] + E[h(\underline{X})].$$

* Example. Let Y = sum of the scores of two fair dice.

Then $Y = \underline{X} + n + 1$ where $\underline{X}$ is

the triangular rv and $n = 6$.

$$E[Y] = E[\underline{X}] + 6 + 1 = 0 + 6 + 1 = 7.$$