The Binomial Distribution

__Definition.__ A binomial experiment consists of a sequence of n independent Bernoulli trials with $P(S) = p$.

The binomial random variable X associated with this experiment is defined as

$$X = \text{the number of } S\text{'s among } n \text{ trials.}$$

__Notation:__ $X \sim Bin(n,p)$.

__Question:__ We would like to know the probability $p(k)$ of exactly k successes. (the pmf of X)

$$p(k) = P(X = k), \quad 0 \le k \le n.$$ integer

(Binomial distribution)

__Notation:__ Because the pmf $p(k)$, $k = 0, 1, \dots, n$, of X depends on the two parameters n and p, we denote the pmf $p(k)$ by $b(k; n, p)$.

__Examples.__ 1°/ A coin is flipped n times. What is the chance of exactly k heads?

2°/ You have n chips. What is the chance that k are defective?

3°/ You buy n lottery scratch cards. What is the chance of k wins?

4°/ You type a page of n symbols. What is the chance of r errors?

...

<u>Theorem</u> (Binomial distribution of $X \sim Bin(n,p)$).

$$b(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \cdots, n$$

Proof. When we perform n Bernoulli trials there are exactly 2^n possible outcomes, because each trial yields either S or F. How many of these outcomes comprise exactly k successes and $n-k$ failures? The answer is

$$\binom{n}{k},$$

because this is the number of distinct ways of ordering k success and $n-k$ failures. Now we observe that, by independence, any given outcome with k successes and $n-k$ failures has probability

$$p^k (1-p)^{n-k}.$$

Hence

$$b(k; n, p) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \left\{ \begin{matrix} \text{number of outcomes} \\ \text{consisting of exactly } k \text{ successes} \\ \text{(and } n-k \text{ failures)} \end{matrix} \right\} \times \left\{ \begin{matrix} \text{probability of} \\ \text{any particular} \\ \text{such outcome} \end{matrix} \right\}.$$

Remark The Binomial distribution is indeed a probability distribution, because

by the binomial theorem

$$\sum_{k=0}^{n} b(k;n,p) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = [p + (1-p)]^n$$

$$= 1.$$

Properties of $b(k;n,p)$

1°/ the recursion:

$$b(k+1;n,p) = \binom{n}{k+1} p^{k+1} (1-p)^{n-k-1}$$

$$= \frac{n-k}{k+1} \left(\frac{n!}{k!\,(n-k)!} \right) \left(\frac{p}{1-p} \right) p^k (1-p)^{n-k}$$

$$= \frac{n-k}{k+1} \left(\frac{p}{1-p} \right) b(k;n,p)$$

2°/ initialization

$$b(0;n,p) = (1-p)^n, \qquad b(n;n,p) = p^n$$

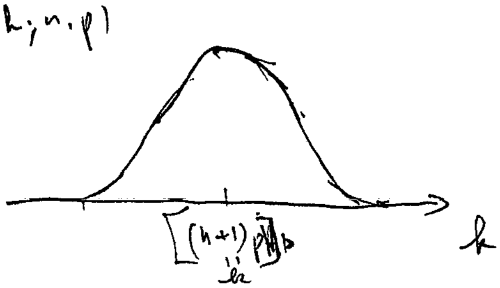
Starting with $b(0;n,p)$ or $b(n;n,p)$, we can use the recursive relation to carry out explicity calculations.

3°/ shape of Binomial distribution.

histogram: $b(k;n,p)$

Note that

$$\frac{b(k;n,p)}{b(k+1,n,p)} = \frac{k+1}{n-k} \left(\frac{1-p}{p} \right) = \begin{cases} < 1, & \forall k < (n+1)p-1 \\ > 1, & \forall k > (n+1)p+1 \\ 1, & \text{if } k = [(n+1)p] \end{cases}$$



Binomial mean of $X \sim Bin(n,p)$: $b(k;n,p)$

$$\mu = E(X) = \sum_{k=0}^{n} k\, p(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \underbrace{\sum_{l=0}^{n-1} \frac{(n-1)!}{l!\,(n-1-l)!} p^{l} (1-p)^{n-1-l}}$$

binomial theorem $\rightsquigarrow$

$$= np\,[p + (1-p)]^{n-1}$$

$$= np$$

$$\left| \; \sum_{l=0}^{n-1} b(l;\, n-1,\, p) = 1 \right.$$

Moment generating function of $X \sim Bin(n,p)$

$$M(t) = \sum_{k=0}^{n} e^{kt} \overset{b(k;n,p)}{p(k)} = \sum_{k=0}^{n} e^{kt} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k}$$

$$= (pe^t + 1 - p)^n$$

Rmk. The binomial theorem is:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Note that

$$M(0) = \sum_{k=0}^{n} p(k) = (p e^{t=0} + 1 - p)^n = 1.$$

The mean and variance of X can be obtained by differentiating $M(t)$:

$$M^{(1)}(t) = n(pe^t + 1 - p)^{n-1} pe^t$$

$$\Rightarrow \mu = M^{(1)}(0) = np$$

$$M^{(2)}(t) = n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)(pe^t) + n(pe^t + 1 - p)^{n-1} pe^t$$

$$\Rightarrow \underset{\substack{\parallel \\ E(X^2)}}{M^{(2)}(0)} = n(n-1)p^2 + np$$

Therefore

$$\sigma^2 = E(X^2) - (E[X])^2 = n(n-1)p^2 + np - n^2p^2$$

$$= np - np^2 = np(1-p).$$

Using Binomial Tables:

Appendix Table A.1 tabulates the cdf $F(k) = P(X \le k)$ for $n = 5, 10, 15, 20, 25$ in combination with selected values of p. The cdf of $X \sim Bin(n, p)$ is denoted by

$k = 0, 1, \dots, n,$ $$B(k; n, p) = P(X \le k) = \sum_{l=0}^{k} b(l; n, p)$$

Examples Let X = the nb of among 15 copies that faid the test with $P(\text{failed}) = 0.2$.

$$\sim Bin(n = 15, p = 0.2)$$

1°/ Then the probability of at most 8 fail test is

$$P(X \le 8) = \underbrace{B(8; 15, 0.2)}_{\text{Table A.1}} = 0.999$$

2°/. The probability of exactly 8 failed test is

$$P(X = 8) = P(x \le 8) - P(x \le 7)$$
$$= B(8; 15, 0.2) - B(7; 15, 0.2)$$
$$= 0.999 - 0.996 = 0.03.$$